A G -SIGNATURE THEOREM FOR V -MANIFOLDS

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ABSTRACT. We prove an analogue of the Atiyah-Singer G -signature theorem for an isometry f on a V -manifold.

An n -dimensional V -manifold is a space locally homeomorphic to the orbit space of an orthogonal action on the Euclidean space R^n by a finite group. Kawasaki [7] proved the analogue of the Hirzebruch signature theorem for V -manifolds. In this paper we will show that the argument in [7] can be modified slightly to give us an analogue of the Atiyah-Singer G -signature theorem [3] for an isometry f on a V -manifold.

1. We recall here some definitions concerning C^∞ V -manifolds, and refer the readers to [4], [8] and [9] for details.

Let G_U be a finite group acting linearly on R^n with the fixed point set of dimension at most $n - 2$, $U^\#$ an invariant open subset of R^n and $q_U: U^\# \rightarrow U = U^\# / G_U$ the projection map satisfying $q_U \circ g = q_U$ for all $g \in G_U$. Then we call U a local V -manifold. A V -map f from one V -manifold U to another W is a (G_U, G_W) -equivariant map $f^\#$ from $U^\#$ to $W^\#$; that is, there exists a homomorphism $\lambda: G_U \rightarrow G_W$ such that $\lambda(g) \circ f^\# = f^\# \circ g$ for $g \in G_U$.

A V -manifold structure on a Hausdorff paracompact space X consists of an open covering X by local V -manifolds $\{U\}$ such that the overlap maps are V -diffeomorphisms. Each $\{U^\#, G_U\}$ (or $\{U^\#, G_U, q_U\}$) is called a local uniformizing system (l.u.s.) for U and G_U a local group [9]. A C^∞ -map is given by a compatible family of equivariant maps on l.u.s.'s. We also have the notion of V -bundle map [9]. In particular, using the differentials of the elements in G_U , we may construct the tangent V -bundle TX .

From now on, we assume that all the V -manifolds are orientable, that is, all the elements of G_U and the overlap maps preserve the orientations.

A Riemannian metric can be introduced on X , given locally by a G_U -invariant metric on $U^\#$ [4, p. 864]. The Chern-Weil construction of the characteristic classes also works in this category [7].

Let w be an n -form on X . Then we define the integral $\int_X w$ via a partition of unity $\{\varphi_U\}$ as follows:

$$\int_X w = \sum_{\{U\}} |G_U|^{-1} \int_{U^\#} \varphi_U^\# w_U^\#$$

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(for an object on U , we use the superscript $\sharp$ to denote its corresponding object on $U^\sharp$).

If $p^\sharp \in U^\sharp$ satisfying $q_U(p^\sharp) = p$, then the isotropy subgroup of G_U at $p^\sharp$ depends only on p , and is denoted by G_p if no confusion will arise [9]. A point p with $G_p = \{1\}$ is called a regular point, otherwise a singular point. Let ΣX denote the set of all singular points. According to [7, p. 77], we may express ΣX as a union of V -manifolds of lower dimensions as follows: Let U_p be a neighborhood of p with $p^\sharp$ as the origin of $U_p^\sharp$, and $(1) = (h_p^0), \dots, (h_p^r)$ be all the conjugacy classes of G_p . Also, let $Z_{G_p}(h_p^j)$ denote the centralizer of h_p^j in G_p , $(U_p^\sharp)^{h_p^j}$ the set of all elements of $U_p^\sharp$ fixed by h_p^j and K_p^j the kernel of the representation $Z_{G_p}(h_p^j) \rightarrow GL((U_p^\sharp)^{h_p^j})$. Then the V -manifold (of various dimensions) FX is given locally by $FU_p = \coprod_{j=1}^r (U_p^\sharp)^{h_p^j} / Z_{G_p}(h_p^j)$ (disjoint unions) with l.u.s. $\{(U_p^\sharp)^{h_p^j}, Z_{G_p}(h_p^j) / K_p^j\}$. Each connected component FX_λ of FX is a V -manifold, and is assigned a number $m_\lambda = |K_p^\lambda|$, the order of K_p^λ .

2. Let E denote a vector V -bundle over a compact V -manifold X . The set $\Gamma(X, E)$ (or just $\Gamma(E)$) consists of all the C^∞ -sections $h: X \rightarrow E$ defined locally by equivariant sections $h_U^\sharp: U^\sharp \rightarrow E_U^\sharp$. We write E' for the dual of E , and Λ^k the k th exterior power of the cotangent bundle T^*X .

Defined locally as G_U -equivariant objects over each l.u.s. $U^\sharp$, we also have the notions of pseudo-differential operators, symbols, ellipticity, parametrices, and index for an elliptic operator, etc. (see [6] or [7]).

We describe next the kernel of a smoothing operator [6, p. 80], [7, p. 82] $A: \Gamma(X, E) \rightarrow \Gamma(X, F)$. Suppose that locally $K_U^\sharp(x, y) \in \Gamma(U^\sharp \times U^\sharp, F_U^\sharp \times (E_U^\sharp \otimes \Lambda^n))$ is a C^∞ -kernel of the corresponding G_U -invariant operator $A_U^\sharp: \Gamma(U^\sharp, E_U^\sharp) \rightarrow \Gamma(U^\sharp, F_U^\sharp)$. We have $K_U^\sharp(x, y) = (g^{-1}, g^{-1})K_U^\sharp(gx, gy)$. But $K_U^\sharp$ is not $G_U \times G_U$ -invariant, hence does not define a section over $U \times U$. Thus locally, the kernel for A_U is defined to be

$$K_U(x, y) = \sum_{g \in G_U} (g^{-1}, 1)K_U^\sharp(gx, y). \quad (2.1)$$

3. Let f be an orientation-preserving isometry of a compact orientable V -manifold X , that is, $f: X \rightarrow X$ is given locally as a local equivariant isometry $f^\sharp$ from one l.u.s. to another (G_U acts as an isometry on each l.u.s. $U^\sharp$).

We assume that (I) the order of f is relatively prime to the order of the elements in each local group G_U , and none of the orders is even; and (II) the fixed point set of f is an oriented V -submanifold Y in X , and $Y_i = Y \cap FX_i$ is a V -submanifold of FX_i for each i (this follows from the first part of (II) and (I)).

If X is of even dimension, then we may define the G -signature $\text{sign}(f, X)$ as in [3, pp. 578–579]. Also, we may apply the Hodge star operator $*$ [4] to the de Rham complex $(\Lambda^i \otimes \mathbb{C}, d)$ [8] to construct the signature complex $0 \rightarrow \Lambda^+ \xrightarrow{D} \Lambda^- \rightarrow 0$ as in [3]. The same algebraic argument in [3, pp. 578–579] gives us the formula $\text{sign}(f, X) = \text{index}(D)(f)$.

We write f for $f^\#$ on each l.u.s. if no confusion will arise. Since the order of f is relatively prime to the order of $g \in G_U$, we see that $fgx = x \Rightarrow gfx = x \Rightarrow fx = g^{-1}x \Rightarrow gx = x$ and $fx = x$. Therefore, locally we have $\text{Fix}(fg) = \text{Fix}(f) \cap \text{Fix}(g)$.

Recall that locally $FX_j = \text{Fix}(g)$ for some $g \in G_U$. Hence $Y_j^\# = \text{Fix}(fg)$ locally. We let $N_j = N(Y_j^\#) = N(\text{Fix}(fg))$ denote the normal bundle of $Y_j^\# = \text{Fix}(fg)$ in $U^\#$. Then N_j splits into eigenspaces of fg : $N_j = N_{\theta_1} \oplus \cdots \oplus N_{\theta_c}$, $(fg)v = e^{i\theta} \cdot v$ if $v \in N_{\theta}$. These splittings are preserved by the action of $Z_{G_U}(g)$ and the attaching map of N_j . Hence we have a global splitting, denoted also by $N_j = N_{\theta_1} \oplus \cdots \oplus N_{\theta_c}$, for each $j = 1, 2, \dots, c$. Since the order of fg is odd, we have Chern classes $c(N_{\theta}) = \prod_{k=1}^r \coth(x_k + i\theta/2)$.

We may also define the L -class of a real vector V -bundle by the L -polynomial in its Pontrjagin classes. Let $L^{Y_j} = L(TY_j) \prod_{k=1}^r L_{\theta_k}(N_{\theta_k})$.

THEOREM. *Let f be an orientation-preserving isometry on a compact orientable V -manifold X^{2n} with $\text{Fix}(f) = Y$ satisfying the conditions (I) and (II) above. Then we have*

$$\text{Sign}(f, X) = \langle L^Y, [Y] \rangle + \sum_{j=1}^c \frac{1}{m_j} \langle L^{Y_j}, [Y_j] \rangle.$$

PROOF. The argument given in [7] can be adapted here. Let $A_1 = D^*D$ and $A_2 = DD^*$ denote the adjoint of D . Both A_1 and A_2 are self-adjoint, positive definite, elliptic operators (on vector V -bundles) of order 2. Hence the spectrum of A_1 (also A_2) is discrete and positive [6, p. 82]. Let $\zeta_A(z)$ denote the zeta-function of A , and for $u > 0$, $u^z \text{ind}(D) = \zeta_{A_1+u}(z) - \zeta_{A_2+u}(z)$ as in [6] or [7]; we may take $z = 0$ and let u approach 0.

Using the notation in [6, pp. 25, 84] or [7, pp. 79–82], we see that

$$\zeta_A(z)(f) = \int_X d \text{tr}[(f^{-1}, 1)K_z(fx, y)], \quad \text{Re}(2z) < -2n.$$

Let $\{\varphi_U\}$ be a partition of unity. We have from (2.1) that

$$\zeta_A(0)(f) = \sum_U |G_U|^{-1} \sum_{g \in G_U} \int_{U^\#} \varphi_U(x) d \text{tr}[(g^{-1}f^{-1}, 1)K_{z,U}^*(fgx, x)]$$

(see [6, p. 85] or [7]).

Then the argument in [6, p. 85] or [7] gives us

$$\text{ind}(D)(f) = \int_Y d(Z_{A_1}^f - Z_{A_2}^f) + \sum_{j=1}^c \frac{1}{m_j} \int_{Y_j} d(Z_{A_1}^{Y_j} - Z_{A_2}^{Y_j}),$$

where locally $dZ_{A_i}^{Y_j} = dZ_{A_i}^{fg}$, which is defined in [7, p. 80].

By resorting to the local version of the Atiyah-Singer G -signature theorem as in [7] or [2], we may identify the j th term on the right-hand side of the equality above with $\int_{Y_j} L^{Y_j} = \langle L^{Y_j}, [Y_j] \rangle$. Q.E.D.

REMARK. In case f generates a finite cyclic group H , the relation between the above theorem and [7] can be illustrated locally as follows: Let a finite

group $H \times G$ act on a C^∞ -manifold M preserving the orientation. Then it induces an orientation-preserving H -action on the V -manifold M/G . Let $\tilde{A}$ be a $H \times G$ -invariant elliptic operator on M , then it induces an H -invariant elliptic operator A on M/G [1]. The index of A on M/G can be defined as the index of the operator $\tilde{A}$ over M restricted to the space of G -invariant sections. Then it follows from the theorem in [5, p. 76] that for $f \in H$, $\text{ind}_{M/G}(A)(f) = |G|^{-1} \sum_{g \in G} \text{ind}_M(\tilde{A})(fg)$.

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