AN EXAMPLE CONCERNING PARTS AND NEWTONIAN CAPACITY¹

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ABSTRACT. We prove the following theorem: Let ϕ be an admissible function with $\phi(0^+) = 0$ and p a nonnegative integer. Then there is a compact set X in the plane and $x \in X$ such that x is a nonpeak point of $R(X)$,

$$\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \mathcal{C}(A_n(x) \setminus P(x)) < \infty,$$

while $\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \gamma(A_n(x) \setminus X) = \infty$, where $A_n(x) = \{2^{-n-1} < |z - x| < 2^{-n}\}$, $P(x)$ denotes the Gleason part of x , γ the analytic capacity and $\mathcal{C}$ the Newtonian capacity.

Let X be a compact subset of the complex plane. We denote by $R(X)$ the uniform closure of all rational functions having no poles on X . For $x, y \in X$, the Gleason distance is

$$\|y - x\| = \sup\{|f(y) - f(x)| : f \in R(X), \|f\| \leq 1\}.$$

The Gleason part of x is $P(x) = \{\|y - x\| < 2\}$. We set $P(x, a) = \{\|y - x\| < a\}$, the Gleason ball about x with radius a .

In [3], O'Farrell gives a necessary condition in terms of Newtonian capacity $\mathcal{C}$ for a point $x \in X$ to be a nonpeak point of $R(X)$ or $R(X)$ admits a p th order bounded point derivation at x , as follows:

THEOREM. (I) *If x is a nonpeak point of $R(X)$ and $a > 0$ then*

$$\sum 2^n \mathcal{C}(A_n(x) \setminus P(x, a)) < \infty,$$

where $A_n(x) = \{2^{-n-1} < |z - x| < 2^{-n}\}$ and $\mathcal{C}$ denotes Newtonian capacity.

(II) *If $R(X)$ admits a p th order bounded point derivation at x and $a > 0$, then*

$$\sum 2^{(p+1)n} \mathcal{C}(A_n(x) \setminus P(x, a)) < \infty.$$

The converse of (I) is trivial, since if x is a peak point, then $P(x, a) = \{x\}$. O'Farrell [3] also points out that $\mathcal{C}$ may not be replaced by analytic capacity γ in this theorem.

One might conjecture that the converse is true in general. This note is to show this conjecture may not be true if $P(x, a)$ is replaced by $P(x)$.

An admissible function ϕ is a positive nondecreasing function on the interval $(0, \infty)$ such that $\psi(r) = r/\phi(r)$ is also nondecreasing, with $\psi(0^+) = 0$.

Melnikov's theorem (see [2], e.g.) states that a point x in a compact set X is

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a peak point for $R(X)$ if and only if $\sum 2^n \gamma(A_n(x) \setminus X) < \infty$. Hallstrom's criterion [2] generalizes that $R(X)$ admits a p th order bounded point derivation at x if and only if

$$\sum 2^{(p+1)n} \gamma(A_n(x) \setminus X) < \infty.$$

With the notion of an admissible function, $\phi(r) = r^\alpha$, $0 < \alpha < 1$, for instance, Wang [5] and O'Farrell [4] discuss the bounded point derivation of fractional order in terms of analytic capacity. Thus it is not hard to see O'Farrell's theorem in a generalized form: if $\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \gamma(A_n(x) \setminus X) < \infty$ then $\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \mathcal{C}(A_n(x) \setminus P(x, a)) < \infty$ for any admissible function ϕ .

Now we state our theorem in a generalized form.

THEOREM. *Let ϕ be an admissible function with $\phi(0^+) = 0$ and p a nonnegative integer. Then there is a compact set X and $x \in X$ such that x is a nonpeak point of $R(X)$.*

$$\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \mathcal{C}(A_n(x) \setminus P(x)) < \infty,$$

while $\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \gamma(A_n(x) \setminus X) = \infty$.

PROOF. Let $a_n = \frac{3}{4} 2^{-n}$, and choose $\rho_n > 0$ so that $\sum 2^n \rho_n < \infty$, but $\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \rho_n = \infty$. Let D_n be the open disk with center a_n and radius ρ_n . Let I_n be the interval $[a_n - \frac{1}{2} \rho_n, a_n + \frac{1}{2} \rho_n]$. Then $\mathcal{C}(I_n) = 0$ (see [1], [6]). Since Newtonian capacity is monotone, subadditive and regular [6], we may choose an open U_n with $I_n \subset U_n$, $\bar{U}_n \subset D_n$, such that

$$\mathcal{C}(\bar{U}_n) < n^{-2} 2^{-(p+1)n} \phi(2^{-n}).$$

Put $X = \bar{\Delta}(0, 1) \setminus \bigcup_1^\infty U_n$. Then $A_n(0) \setminus X = U_n$, so $\gamma(A_n(0) \setminus X) \leq \gamma(D_n) = \rho_n$, whence $\sum 2^n \gamma(A_n(0) \setminus X) \leq \sum 2^n \rho_n < \infty$, so 0 is a nonpeak point of $R(X)$, by Melnikov's theorem. It follows [3] that the part P of 0 contains the interior of X , so that $A_n(0) \setminus P \subset \bar{U}_n$. Thus

$$\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \mathcal{C}(A_n(0) \setminus P) \leq \sum 2^{(p+1)n} \phi(2^{-n})^{-1} \mathcal{C}(\bar{U}_n) < \sum n^{-2} < \infty,$$

while

$$\begin{aligned} \sum 2^{(p+1)n} \phi(2^{-n})^{-1} \gamma(A_n(0) \setminus X) &\geq \sum 2^{(p+1)n} \phi(2^{-n})^{-1} \gamma(I_n) \\ &= \frac{1}{4} \sum 2^{(p+1)n} \phi(2^{-n})^{-1} \rho_n = \infty \end{aligned}$$

(see [1]).

REMARK. It can be shown that the series $\sum 2^{(p+1)n} \phi(2^{-n})^{-1} \mathcal{C}(A_n(0) \setminus P(0, a))$ diverges for any a , $0 < a < 2$, for all compact sets X constructed in the theorem. So to give a counterexample to the converse of O'Farrell's result for $a < 2$, it appears that some other compact set X is needed. We are grateful to Bernt Øksendal for his helpful communications.

REFERENCES

1. J. Garnett, *Analytic capacity and measure*, Lecture Notes in Math., vol. 297, Springer-Verlag, Berlin, 1972.
2. A. Hallstrom, *On bounded point derivations and analytic capacity*, J. Functional Analysis 4 (1969), 153–165.
3. A. O'Farrell, *Density of parts of algebras on the plane*, Trans. Amer. Math. Soc. 196 (1974), 403–414.
4. ———, *Analytic capacity and equicontinuity* (preprint).
5. J. Wang, *Modules of approximate continuity for $R(X)$* , Math. Scand. 34 (1974), 219–225.
6. J. Wermer, *Potential theory*, Lecture Notes in Math., vol. 408, Springer-Verlag, Berlin, 1974.

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