ON CERTAIN BOUNDED SOLUTIONS OF $g * \mu = f$

A. M. FINK AND W. R. MADYCH

ABSTRACT. We show that, under certain conditions, bounded solutions g of $g * \mu = f$ are asymptotically almost periodic.

1. Introduction. We consider the asymptotic behavior as $t \rightarrow \infty$ of bounded solutions of the linear integral equation

$$g * \mu(t) = f(t), \quad -\infty < t < \infty. \quad (1)$$

Here

$$g * \mu(t) = \int_{-\infty}^{\infty} g(t-s) d\mu(s),$$

where μ is a bounded Borel measure on $R = (-\infty, \infty)$.

In what follows, $\hat{\mu}$ always denotes the Fourier transform of the measure μ , namely

$$\hat{\mu}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} d\mu(t).$$

$\Lambda(\mu) = \{\xi \in R: \hat{\mu}(\xi) = 0\}$ is the set of zeros of $\hat{\mu}$. A function g in $L^\infty = L^\infty(R)$ is said to satisfy *condition T*, if for every positive ε , there is an r and positive δ such that $|g(t) - g(s)| < \varepsilon$ whenever $|t - s| < \delta$ and both t and s are greater than r .

Our first theorem is a generalization of a result due to Beurling [1].

THEOREM 1. *Suppose g is a bounded solution of $g * \mu = 0$. Furthermore, suppose that g satisfies condition T and that $\Lambda(\mu)$ has no finite accumulation point. Then g is equal almost everywhere to an almost periodic function h and $\exp h \subset \Lambda(\mu)$.*

In the case that $\Lambda(\mu) = \emptyset$, the following variant of Pitt's Tauberian theorem holds. (See [5, p. 409].)

THEOREM 2. *Suppose g is a bounded solution of $g * \mu = f$, where*

(i) *f is in L^∞ and $\lim_{t \rightarrow \infty} f(t) = 0$,*

(ii) *g satisfies condition T, and*

(iii) *$\Lambda(\mu) = \emptyset$.*

Then $\lim_{t \rightarrow \infty} g(t) = 0$.

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A combination of Theorems 1 and 2 should result in a general statement concerning asymptotic behavior of solutions of (1). Unfortunately more restrictions on f and μ are needed. Hence the following definitions.

Given a positive integer m , we say that a bounded function f satisfies property $M(f, m)$ if $\lim_{t \rightarrow \infty} f(t) = 0$ and $\int_0^\infty t^{m-1} |f(t)| dt$ is finite. The integer m may be taken to be ∞ ; in this case the above integral must be finite for all positive integers m .

If μ is a bounded Borel measure, we say that μ satisfies property $M\Lambda(\mu)$ if for every $\xi_j \in \Lambda(\mu)$ there exists a positive integer m_j such that the function $\hat{\nu}_j$, defined by $\hat{\nu}_j(\xi) = (\xi - \xi_j)^{-m_j} \hat{\mu}(\xi)$, is the Fourier transform of a bounded measure and $\hat{\nu}_j(\xi_j) \neq 0$. Clearly any μ for which $\Lambda(\mu) = \emptyset$ satisfies property $M\Lambda(\mu)$.

THEOREM 3. Suppose g is a bounded solution of $g * \mu = f$, where

- (i) $\Lambda(\mu)$ has no finite accumulation point,
- (ii) μ satisfies property $M\Lambda(\mu)$,
- (iii) if $m = \max\{m_j\}$, where m_j was defined in the above paragraph, then f satisfies property $M(f, m)$,
- (iv) g satisfies property T .

Then $g = g_1 + g_2$, where g_1 is bounded and $\lim_{t \rightarrow \infty} g_1(t) = 0$ and g_2 is almost periodic with $\exp g_2 \subset \Lambda(\mu)$.

This theorem is an improvement of a result due to Jordan and Wheeler [3]. It should be noted that property $M\Lambda(\mu)$ is somewhat less stringent than the condition on certain moments of μ found there.

It appears that it may be possible to relax conditions (i) and (ii) somewhat (i.e., fractional m_j 's, unbounded μ , etc.). However (iii) and (iv) cannot be significantly relaxed using the methods employed below.

2. Details. First we recall some standard notation: $\mathcal{S} = \mathcal{S}(R)$ is the Schwartz space, namely the space of rapidly decreasing complex valued infinitely differentiable functions on R together with its topology. $\mathcal{S}'$ is its dual, the space of tempered distributions. Given an element f in $\mathcal{S}'$, $\hat{f}$ denotes its Fourier transform. If $\mathcal{Q}$ is a subspace of $\mathcal{S}'$ then $\hat{\mathcal{Q}} = \{\hat{f}: f \in \mathcal{Q}\}$. $\mathfrak{M}$ denotes that subspace of $\mathcal{S}'$ whose elements are bounded Borel measures. If A is an open subset of R then $\mathcal{D}(A)$ is the space of infinitely differentiable functions with support in A . L^p , $1 \leq p < \infty$, denotes the usual space of functions whose p th power is absolutely summable, $\|\cdot\|_p$ denotes the L^p norm.

$f \in \text{a.p.}$ means that f is almost periodic. Basic facts concerning almost periodic functions used here can be found in Chapter 1 of [2]. If $f \in \text{a.p.}$ then $\exp f$ denotes the set of exponentials of f .

The symbol ψ always denotes a function in $\hat{\mathcal{D}}(R)$ whose Fourier transform $\hat{\psi}$ has the following properties: $\hat{\psi}$ is nonnegative, $\hat{\psi}(\xi) = 1$ if $|\xi| < 1$ and

$\hat{\psi}(\xi) = 0$ if $|\xi| > 2$. If r is a positive number ψ_r is defined by $\psi_r(t) = r\psi(rt)$. Recall that $\hat{\psi}_r(\xi) = \hat{\psi}(\xi/r)$.

The symbol C will be used generically for constants appearing in certain estimates and need not be the same at different occurrences.

The following lemmas will be used repeatedly in the proofs of the above theorems.

LEMMA 1. Suppose that $f \in a.p.$ and, for some positive ε , $|f(t)| < \varepsilon$ for $t > r$. Then $\|f\|_\infty < \varepsilon$.

To see this, let δ be any positive number and let $T(f, \delta)$ denote the δ -translation set of f . If t is any point in R , using the fact that $T(f, \delta)$ is relatively dense, it follows that $[r - t, \infty) \cap T(f, \delta) \neq \emptyset$. Hence if s is any point of $[r - t, \infty) \cap T(f, \delta)$ then $t + s > r$ and hence $|f(t)| < |f(t + s)| + |f(t) - f(t + s)| < \varepsilon + \delta$. Since δ was an arbitrary positive number the lemma follows.

LEMMA 2. Suppose g is a bounded function which satisfies condition T . Then

- (i) $\lim_{r \rightarrow \infty} \psi_r * g = g$ almost everywhere, and
- (ii) for any positive number ε , there is an r_0 and t_0 , such that $|\psi_r * g(t) - g(t)| < \varepsilon$ whenever $r > r_0$ and $t > t_0$.

The first fact is classical and a detailed proof may be found in [7]. The second follows from a routine calculation. For the sake of completeness we recall it. Write

$$\psi_r * g(t) - g(t) = \int_{|s| < \delta} \psi_r(s) [g(t - s) - g(t)] ds + \int_{|s| > \delta} \psi_r(s) [g(t - s) - g(t)] ds.$$

Now for r sufficiently large, the absolute value of the second integral is dominated by $2\varepsilon_0 \|g\|_\infty$. If δ is sufficiently small and t sufficiently large, using condition T , the absolute value of the first integral can be dominated by $\varepsilon_0 \|\psi\|_1$. Choosing $\varepsilon_0 = \varepsilon / (\|\psi\|_1 + 2\|g\|_\infty)$ results in the desired conclusion.

LEMMA 3. Suppose k is in $\mathfrak{S}$ and f is in L^∞ . Then $k * f \in L^\infty$ and

- (i) if $\lim_{t \rightarrow \infty} f(t) = 0$, then $\lim_{t \rightarrow \infty} k * f(t) = 0$,
- (ii) if f satisfies condition $M(f, m)$, then $k * f$ satisfies condition $M(k * f, m)$.

The fact that $k * f$ is in L^∞ and (i) is true is easy to verify. To see (ii), write

$$\begin{aligned} \int_0^\infty t^{m-1} |k * f(t)| dt &< \int_{-\infty}^\infty |k(s)| \int_0^\infty t^{m-1} |f(t - s)| dt ds \\ &= \int_{-\infty}^0 |k(s)| \int_{-s}^\infty (u + s)^{m-1} |f(u)| du ds \\ &\quad + \int_0^\infty |k(s)| \int_{-s}^\infty (u + s)^{m-1} |f(u)| du ds \\ &= I_1 + I_2. \end{aligned}$$

First,

$$I_1 \leq C \int_{-\infty}^0 |k(s)| \left(\int_0^{\infty} u^{m-1} |f(u)| du + |s|^{m-1} \int_0^{\infty} |f(u)| du \right) ds,$$

and it is clear that I_1 is finite. Now

$$\begin{aligned} I_2 &\leq C \int_0^{\infty} |k(s)| \left(\int_{-s}^0 (|u|^{m-1} + |s|^{m-1}) |f(u)| du \right. \\ &\quad \left. + \int_0^{\infty} (|u|^{m-1} + |s|^{m-1}) |f(u)| du \right) ds \\ &\leq C \int_0^{\infty} |k(s)| (s^m + 1 + s^{m-1}) ds. \end{aligned}$$

The last inequality shows that I_2 is also finite and hence the proof of this lemma is complete.

LEMMA 4. Suppose ϕ is a bounded continuous function on R such that for some $\xi_1 < \xi_2$, $\phi(\xi_1) \neq 0$ and $\phi(\xi_2) \neq 0$. Then there is a θ in $\mathfrak{N}$ such that

- (i) $\hat{\theta}(\xi) + \phi(\xi) \neq 0$ for $\xi \in (-\infty, \xi_1] \cup [\xi_2, \infty)$,
- (ii) $(\xi - \xi_0)^{-m} \hat{\theta}(\xi)$ is in $\mathfrak{N}$ for $m = 1, 2, \dots$ and all ξ_0 satisfying $\xi_1 < \xi_0 < \xi_2$.

PROOF. First, assume that the real parts of $\phi(\xi_1)$ and $\phi(\xi_2)$ are both positive. Then there is a positive ε such that the real part of $\phi(\xi)$ is positive for ξ in $(\xi_1 - \varepsilon, \xi_1) \cup (\xi_2, \xi_2 + \varepsilon)$. Let f be an infinitely differentiable function with the following properties: $f(\xi) = 0$ for $\xi_1 < \xi < \xi_2$, $f(\xi) > 0$ for ξ in the complement of $[\xi_1, \xi_2]$, and $f(\xi) = 2\|\phi\|_{\infty}$ in the complement of $[\xi_1 - \varepsilon, \xi_2 + \varepsilon]$. Clearly $f(\xi) + \phi(\xi) \neq 0$ if $\xi \in (-\infty, \xi_1] \cup [\xi_2, \infty)$. Furthermore $f = \text{const} + g$, where g is in $\mathcal{D}(R)$. So $\hat{f} = \mu + \hat{g}$ where μ is a constant multiple of the Dirac measure and $\hat{g}$ is in $\mathfrak{S}(R)$ and hence, f is in $\mathfrak{N}$. Direct calculations or an application of Theorem 3 in [4] show that $(\xi - \xi_0)^{-m} f(\xi)$ is in $\mathfrak{N}$ for all positive integers m and all ξ_0 in (ξ_1, ξ_2) . Hence $\hat{\theta} = f$ does the job in this case.

If $\text{Re } \phi(\xi_1) > 0$ and $\text{Re } \phi_2(\xi_2) < 0$, then the real parts of

$$\phi_1(\xi) = \phi(\xi) \exp(i\pi(\xi_2 - \xi_1)^{-1}(\xi - \xi_1))$$

evaluated at ξ_1 and ξ_2 are both positive. It should be clear that in this case

$$\hat{\theta}(\xi) = f(\xi) \exp(i\pi(\xi_2 - \xi_1)^{-1}(\xi_1 - \xi))$$

does the job, where f is defined as above. The remaining cases are handled similarly.

PROOF OF THEOREM 1. Let $\hat{\phi}$ be any function in $\mathcal{D}(R \setminus \Lambda(\mu))$. There exists a function f such that $f * \mu = \phi$ (i.e., $\hat{f} = \hat{\phi}/\hat{\mu}$ is in $\hat{L}^1$ due to a well-known variant of one of Wiener's theorems; for example, see [6]). Convolving f with both sides of $g * \mu = 0$ results in $\phi * g = 0$ or $\hat{\phi}\hat{g} = 0$. It follows that the support of $\hat{g}$ is contained in Λ . If $\xi_i \in \Lambda$, choose r_j sufficiently small so that the support $\hat{\psi}_{j,r_j}(\xi) = \hat{\psi}((\xi - \xi_j)/r_j)$ contains only one element of $\Lambda(\mu)$,

namely, ξ_j . The support of $\hat{g}\hat{\psi}_{j,r_j}$ is contained in $\{\xi_j\}$ and hence $\hat{g}\hat{\psi}_{j,r_j}$ is a finite linear combination of the Dirac measure supported on $\{\xi_j\}$ and its derivatives. Since the boundedness of g implies the boundedness of $g * \psi_{j,r_j}$, it follows that $g * \psi_{j,r_j} = c_j e^{i\xi_j t}$ for some constant c_j . Now it should be clear that $g * \psi_n = \sum c_j e^{i\xi_j t}$, where the c_j 's are finite constants and the sum is finite, namely, it is taken over those j 's such that ξ_j is contained in the interval $(-2n, 2n)$.

From Lemma 2 it follows that for any positive ε , if t and N are sufficiently large then $|g * \psi_m(t) - g * \psi_n(t)| < \varepsilon$ whenever m and n are both greater than N . Since $g * \psi_m - g * \psi_n$ is almost periodic, it follows from Lemma 1 that $\|g * \psi_m - g * \psi_n\|_\infty < \varepsilon$ and hence we conclude that $g * \psi_n$ converges uniformly to an almost periodic function h with $\exp h \subset \Lambda(\mu)$. But $g * \psi_m$ converges almost everywhere to g . Hence $g = h$ and the theorem follows.

PROOF OF THEOREM 2. Let ϕ be the function defined by $\phi(t) = \exp(-t^2)$. Convolve both sides of $g * \mu = f$ with ϕ to get $g * \mu * \phi = f * \phi$. Clearly $f * \phi(t) \rightarrow 0$ as $t \rightarrow \infty$, $\mu * \phi$ is in L^1 and $\Lambda(\mu * \phi) = \emptyset$. It follows, again from one of Wiener's theorems, that finite linear combinations of $\mu * \phi$ are dense in L^1 . If K denotes such a finite linear combination of translates of $\mu * \phi$, then $g * K =$ same finite linear combination of translates of $f * \phi$ and hence $g * K(t) \rightarrow 0$ as $t \rightarrow \infty$. Now for any positive ε , let N_1 , r , and K be such that $|g(t) - \psi_r * g(t)| < \varepsilon$ whenever $t > N_1$ and $\|\psi_r - K\|_1 < \varepsilon$. If N_2 is such that $|g * K(t)| < \varepsilon$ if $t > N_2$, then

$$\begin{aligned} |g(t)| &\leq |g(t) - \psi_r * g(t)| + |\psi_r * g(t) - g * K(t)| + |g * K(t)| \\ &\leq \varepsilon + \varepsilon \|g\|_\infty + \varepsilon \end{aligned}$$

whenever $t > \max\{N_1, N_2\}$. The theorem now follows from the last inequality involving g .

PROOF OF THEOREM 3. We break up the proof into three cases:

(i) $\Lambda(\mu)$ contains only one point, ξ_1 , (ii) $\Lambda(\mu)$ is a finite set, and (iii) $\Lambda(\mu)$ is infinite.

Case (i). As in the proof of Theorem 2, let ϕ be defined by $\hat{\phi}(\xi) = \exp(-\xi^2)$ and convolve both sides of $g * \mu = f$ with ϕ . Taking Fourier transforms we get $(\xi - \xi_1)^{m-1} (g * \phi * \nu_1)^\wedge = (\phi * f)^\wedge$ where ν_1 is the measure whose Fourier transform is $(\xi - \xi_1)^{-m} \hat{\mu}(\xi)$ and $\Lambda(\nu_1) = \emptyset$. Since $g * \phi * \nu_1$ is infinitely differentiable, applying the inverse Fourier transform to both sides of the last equation results in

$$D^m (e^{-i\xi_1 t} g * \phi * \nu_1(t)) = e^{-i\xi_1 t} \phi * f(t),$$

where D^m is the differential operator d^m/dt^m . According to Lemma 3, $\phi * f$ satisfies $M(\phi * f, m)$ so

$$e^{-i\xi_1 t} g * \phi * \nu_1(t) = (-1)^m \int_t^\infty \frac{(s-t)^{m-1}}{(m-1)!} e^{-i\xi_1 s} \phi * f(s) ds + \sum_{j=0}^{m-1} c_j t^j.$$

Since the left-hand side of the above equality is bounded and the first term on the right-hand side goes to 0 as $t \rightarrow \infty$, it follows that $c_1 = \cdots = c_{m-1} = 0$ and hence

$$\begin{aligned} g * \phi * \nu_1(t) &= (-1)^m e^{i\xi_1 t} \int_t^\infty \frac{(s-t)^{m-1}}{(m-1)!} e^{-i\xi_1 s} \phi * f(s) ds + c_0 e^{i\xi_1 t} \\ &= h_1(t) + h_2(t). \end{aligned}$$

Given a positive ε , let N_1 and N_2 be such that $|g * \psi_n(t) - g * \psi_m(t)| < \varepsilon$ whenever both n and m are greater than N_1 and $t > N_2$. If m and n are both $> N_1$, let K_m and K_n be finite linear combinations of translates of $\phi * \nu_1$ such that $\|\psi_j - K_j\|_1 < \varepsilon$, $j = m$ and n . Now $g * K_j(t) = h_1^j(t) + h_2^j(t)$ where h_1^j and h_2^j are finite linear combinations of translates of h_1 and h_2 , respectively, $j = m$ and n .

Let N_3 be such that $|h_1^j(t)| < \varepsilon$ whenever $t > N_3$. Now

$$\begin{aligned} |h_2^n(t) - h_2^m(t)| &< |g * K_n(t) - g * K_m(t)| + |h_1^n(t)| + |h_1^m(t)| \\ &< |g * \psi_n(t) - g * \psi_m(t)| + |g * K_n(t) - g * \psi_n(t)| \\ &\quad + |g * \psi_m(t) - g * K_m(t)| + |h_1^n(t)| + |h_1^m(t)|. \end{aligned}$$

Hence if $t > \max\{N_2, N_3\}$ it follows that

$$|h_2^n(t) - h_2^m(t)| < 4\varepsilon + 2\varepsilon \|g\|_\infty.$$

It follows from Lemma 1 that $\|h_2^n - h_2^m\|_\infty < C\varepsilon$, where C is a constant independent of m and n if both m and n are $> N_1$. Now

$$g = g - g * \psi_n + g * \psi_n - g * K_n + h_1^n(t) + h_2^n(t)$$

where $g - g * \psi_n \rightarrow 0$ almost everywhere as $n \rightarrow \infty$, $g * \psi_n - g * K_n \rightarrow 0$ uniformly as $n \rightarrow \infty$ and $h_2^n \rightarrow g_2(t)$ uniformly where $g_2(t) = C_\infty e^{i\xi_1 t}$. It follows that h_1^n converges to a bounded function g_1 . To see that $\lim_{t \rightarrow \infty} g_1(t) = 0$, write

$$\begin{aligned} g_1(t) - h_1^n(t) &= g(t) - g * K_n(t) - g_2(t) + h_2^n(t) \\ &= g(t) - g * \psi_n(t) + g * \psi_n(t) - g * K_n(t) - g_2(t) + h_2^n(t). \end{aligned}$$

Now for n sufficiently large $g * \psi_n - g * K_n$ and $g_2 - h_2^n$ are uniformly small, and if t is sufficiently large then $g(t) - g * \psi_n(t)$ is small. Hence if t and n are sufficiently large $g_1(t) - h_1^n(t)$ is small. Since

$$|g_1(t)| < |g_1(t) - h_1^n(t)| + |h_1^n(t)|,$$

choosing n and t sufficiently large makes $|g(t)|$ small.

Case (ii). Let $\Lambda(\mu) = \{\xi_1, \dots, \xi_n\}$, where $\xi_1 < \xi_2 < \cdots < \xi_n$. Choose ξ_0 and ξ_{n+1} such that $\xi_0 < \xi_1$ and $\xi_n < \xi_{n+1}$ and let $\hat{\phi}_1, \dots, \hat{\phi}_n$ be nonnegative functions in $\mathcal{D}(R)$ such that (a) the intersection of the support of $\hat{\phi}_j$ with $\Lambda(\mu)$ contains only ξ_j , $j = 1, \dots, n$, (b) the support of $\hat{\phi}_j$ is contained in a "small" neighborhood of ξ_j , and (c) $\sum \hat{\phi}_j(\xi) = 1$ if $\xi_0 < \xi < \xi_{n+1}$. Let $\hat{\phi}_0 = 1 - \sum \hat{\phi}_j$ and let $\hat{\theta}_0, \dots, \hat{\theta}_n$ be functions in $\mathcal{M}$ whose support is contained in the

complement of the support of $\hat{\phi}_j$, $j = 0, \dots, n$, respectively, and such that $\Lambda(\mu + \theta_0) = \emptyset$, $\Lambda(\mu + \theta_j) = \xi_j$, $j = 1, \dots, n$, and the $\mu + \theta_j$ satisfy property $M\Lambda(\mu + \theta_j)$, $j = 0, \dots, n$. That such θ 's exist is a consequence of Lemma 4.

Since $\phi_j * g * \theta_j = 0$, it follows that $\phi_j * g * (\mu + \theta_j) = \phi_j * f$, $j = 0, 1, \dots, n$. Applying the result of case (i), it follows that $\phi_j * g = h_1^j + h_2^j$, where h_1^j is bounded and $\lim_{t \rightarrow \infty} h_1^j(t) = 0$, $j = 0, 1, \dots, n$, and $h_2^0 = 0$ and $h_2^j(t) = C_j e^{i\xi_j t}$, $j = 1, \dots, n$. Since $g = \sum_{j=0}^n \phi_j * g$, it follows that $g = g_1 + g_2$ as advertised.

Case (iii). Choose an increasing sequence of positive numbers, $\{r_j\}_{j=1}^\infty$, such that

$$(\{2r_j\}_{j=1}^\infty \cup \{-2r_j\}_{j=1}^\infty) \cap \Lambda(\mu) = \emptyset.$$

Let $\hat{\theta}_j$ be functions in $\mathfrak{M}$ whose support is contained in $\{\xi: |\xi| > 2r_j\}$ and such that $\hat{\mu}(\xi) + \hat{\theta}_j(\xi) \neq 0$ for $|\xi| > 2r_j$. Since $\psi_{r_j} * g * \theta_j = 0$, we have $\psi_{r_j} * g * (\mu + \theta_j) = \psi_{r_j} * f$. Since $\Lambda(\mu + \theta_j)$ is a finite set, and θ_j may be chosen so that $\mu + \theta_j$, $\psi_{r_j} * g$, and $\psi_{r_j} * f$ satisfy the hypothesis on μ , g , and f of Theorem 3, it follows from case (ii) that $\psi_{r_j} * g = h_1^j + h_2^j$, where h_1^j is bounded and $\lim_{t \rightarrow \infty} h_1^j(t) \rightarrow 0$ and h_2^j is almost periodic, $j = 1, 2, \dots$. Now, an argument similar to that used in case (i) shows that $g = h_1 + h_2$ as in the statement of the theorem.

This completes the proof of Theorem 3.

3. Remarks. It was mentioned in the introduction that Theorem 3 is an improvement of the result of Jordan and Wheeler [3] even in the case when $\Lambda(\mu)$ is a finite set. This can be seen from the following:

LEMMA 5. Suppose μ is in $\mathfrak{M}$ and satisfies

$$\int_{-\infty}^{\infty} |t|^m |d\mu(t)| < \infty,$$

$$\int_{-\infty}^{\infty} e^{-i\xi_0 t} t^j d\mu(t) = 0, \quad j = 0, \dots, m-1,$$

and

$$\int_{-\infty}^{\infty} e^{-i\xi_0 t} t^m d\mu(t) \neq 0.$$

Then the function $\hat{v}$ defined by $\hat{v}(\xi) = (\xi - \xi_0)^{-m} \hat{\mu}(\xi)$ is in $\mathfrak{M}$ and $\hat{v}(\xi_0) \neq 0$.

PROOF. Without loss of generality we may take $\xi_0 = 0$. Let

$$f(t) = \int_{-\infty}^t \{(t-s)^{m-1} / (m-1)!\} d\mu(s).$$

Note that f is in L^1 . To see this, use the fact that certain moments of μ are zero and write

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(t)| dt &\leq \int_{-\infty}^0 \int_{-\infty}^t \frac{(t-s)^{m-1}}{(m-1)!} |d\mu(s)| dt \\
&\quad + \int_0^{\infty} \int_t^{\infty} \frac{(s-t)^{m-1}}{(m-1)!} |d\mu(s)| dt \\
&\leq \int_{-\infty}^0 \frac{|s|^m}{m!} |d\mu(s)| + \int_0^{\infty} \frac{s^m}{m!} |d\mu(s)| < \infty.
\end{aligned}$$

Since f is m times differentiable and its m th derivative is the measure μ it is clear that $(i\xi)^m \hat{f}(\xi) = \hat{\mu}(\xi)$, or $\hat{f}(\xi) = (i\xi)^{-m} \hat{\mu}(\xi)$. Now clearly

$$\hat{f}(0) = \int_{-\infty}^{\infty} \int_{-\infty}^t \frac{(t-s)^{m-1}}{(m-1)!} d\mu(s) dt.$$

Breaking up the integral as in the previous calculation we see that

$$\hat{f}(0) = \frac{1}{m!} \int_{-\infty}^{\infty} s^m d\mu(s) \neq 0.$$

Since $\hat{\nu}(\xi)$ is simply a constant multiple of $\hat{f}$, the proof of the lemma is complete.

A moment's reflection shows that the "moment" condition $M\Lambda(\mu)$ is considerably weaker than the condition $H(\mu, m)$ in [3].

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DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50011