INTEGRAL REPRESENTATIONS AND THE COMPLEX MONGE-AMPÈRE EQUATION IN STRICTLY CONVEX DOMAINS

NANCY K. STANTON¹

ABSTRACT. We prove a relationship between Aizenberg's integral representation formula for holomorphic functions in a strictly convex domain and the complex Monge-Ampère equation.

Let Ω be a bounded strictly convex domain in $\mathbb{C}^n$, $n > 1$, with a smooth boundary. Let r be a $\mathcal{C}^\infty$ defining function for Ω , so $\Omega = \{z: r(z) < 0\}$ and $dr \neq 0$ on $b\Omega$. If $z \in \Omega$ and $\zeta \in b\Omega$ then $\sum r_i(\zeta)(\zeta_i - z_i) \neq 0$, since Ω lies on one side of the tangent space to $b\Omega$ at ζ . Here $r_i = \partial r / \partial z_i$, $r_{\bar{i}} = \partial r / \partial \bar{z}_i$, etc.

Aizenberg [2] proved that if f is holomorphic on Ω and continuous on $\bar{\Omega}$, one has the following integral representation formula for f :

$$f(z) = \left(\frac{1}{2\pi i} \right)^n (-1)^{(n-1)(n-2)/2} \int_{b\Omega} \frac{f(\zeta) \partial r(\zeta) \wedge (\partial \bar{\partial} r(\zeta))^{n-1}}{(\sum r_i(\zeta)(\zeta_i - z_i))^n}. \quad (1)$$

(This way of writing the formula is due to Norguet [4, Theorem 11].) The expression

$$K_r(\zeta, z) = \frac{\partial r(\zeta) \wedge (\partial \bar{\partial} r(\zeta))^{n-1}}{(\sum r_i(\zeta)(\zeta_i - z_i))^n} \quad (2)$$

is an example of a Cauchy-Fantappiè form [4].

Let $J(r)$ denote the Levi determinant,

$$J(r) = (-1)^n \det \begin{bmatrix} r & r_{\bar{j}} \\ r_i & r_{i\bar{j}} \end{bmatrix}.$$

THEOREM 1. Let $d\sigma$ denote the induced volume element on $b\Omega$. Then formula (1) can be rewritten as

$$f(z) = (-1)^{(n-1)(n-2)/2} \frac{(n-1)!}{\pi^n} \int_{b\Omega} \frac{f(\zeta)(J(r))(\zeta)d\sigma(\zeta)}{|dr(\zeta)|(\sum r_i(\zeta)(\zeta_i - z_i))^n}. \quad (3)$$

REMARK. If the numerator of K_r is expanded as a linear combination of the $d\zeta_i$ and $d\bar{\zeta}_i$, then for $\zeta \in b\Omega$, the coefficients are certain minors of $J(r)$ [2].

PROOF OF THEOREM 1. The restriction of the form $\partial r \wedge (\partial \bar{\partial} r)^{n-1}$ to $b\Omega$ is a multiple of $d\sigma$, $i^*(\partial r \wedge (\partial \bar{\partial} r)^{n-1}) = g d\sigma$, where $i: b\Omega \rightarrow \mathbb{C}^n$ is inclusion. Since $(dr/|dr|) \wedge d\sigma$ is the Euclidean volume element dv at points of $b\Omega$,

Received by the editors August 14, 1978.

AMS (MOS) subject classifications (1970). Primary 53C65, 32A25.

¹Research supported in part by NSF grant MCS 76-08478.

© 1979 American Mathematical Society
0002-9939/79/0000-0317/\$01.75

$$(dr/|dr|) \wedge (\partial r \wedge (\partial \bar{\partial} r)^{n-1}) = (\bar{\partial} r/|dr|) \wedge \partial r \wedge (\partial \bar{\partial} r)^{n-1} = g dv.$$

Let μ be the $2n$ form $\mu = J(r)dv$. We will show that $\bar{\partial} r \wedge \partial r \wedge (\partial \bar{\partial} r)^{n-1} = (2i)^n(n-1)!\mu$. Then $g = (((2i)^n(n-1)!)/|dr|)J(r)$ and formula (3) follows from (1). Fix $\zeta_0 \in b\Omega$ and let ψ be a biholomorphic map defined in a neighborhood of ζ_0 . Since ψ is biholomorphic,

$$\psi^*(\bar{\partial} r \wedge \partial r \wedge (\partial \bar{\partial} r)^{n-1}) = \bar{\partial}(r \circ \psi) \wedge \partial(r \circ \psi) \wedge (\partial \bar{\partial}(r \circ \psi))^{n-1}.$$

A simple calculation shows that $J(r \circ \psi)(\zeta) = |(\partial \psi / \partial z)(\zeta)|^2 J(r)(\psi(\zeta))$. Hence, $\psi^*(\mu(\psi(\zeta))) = (J(r \circ \psi)dv)(\zeta)$. Thus, it suffices to show that, for an appropriate choice of ψ with $\psi(\zeta_0) = \zeta_0$,

$$\begin{aligned} & (\bar{\partial}(r \circ \psi) \wedge \partial(r \circ \psi) \wedge (\partial \bar{\partial}(r \circ \psi))^{n-1})(\zeta_0) \\ &= (2i)^n(n-1)!J(r \circ \psi)(\zeta_0)dv(\zeta_0). \end{aligned} \quad (4)$$

We can choose ψ so that $(r \circ \psi)_i(\zeta_0) = 0$ for $1 \leq i \leq n-1$, $(r \circ \psi)_n(\zeta_0) = a \neq 0$ and $(r \circ \psi)_{ij}(\zeta_0) = \lambda_i \delta_{ij}$ for $1 \leq i, j \leq n-1$. Then the left-hand side of (4) is

$$|a|^2 d\bar{\zeta}_n \wedge d\zeta_n \wedge \left(\sum_{i=1}^{n-1} \lambda_i d\zeta_i \wedge d\bar{\zeta}_i \right)^{n-1} = -(-2i)^n(n-1)!|a|^2 \lambda_1 \cdots \lambda_{n-1} dv.$$

Since $J(r \circ \psi)(\zeta_0) = (-1)^{n-1}|a|^2 \lambda_1 \cdots \lambda_{n-1}$, (4) holds.

COROLLARY 2. *Let r be a defining function for Ω which is an approximate solution to the Monge-Ampère equation, i.e., $J(r) = 1 + O(r^2)$. Then if f is holomorphic on Ω and continuous on $\bar{\Omega}$,*

$$f(z) = (-1)^{(n-1)(n-2)/2} \frac{(n-1)!}{\pi^n} \int_{b\Omega} \frac{f(\zeta) d\sigma(\zeta)}{|dr(\zeta)|(\sum r_i(\zeta)(\zeta_i - z_i))^n}.$$

By Fefferman's work on the Monge-Ampère equation [3], we can always find such an r and it is uniquely determined through first order along $b\Omega$. Approximate solutions to the Monge-Ampère equation are closely related to the geometry of $b\Omega$ [3].

Although making such a choice of r simplifies formula (3), the kernel K_r turns out to be independent of the choice of r .

THEOREM 3. *Let r and r' be defining functions for Ω and let $i: b\Omega \rightarrow \mathbb{C}^n$ be the inclusion map. Then $i_*^* K_r(\zeta, z) = i_*^* K_{r'}(\zeta, z)$.*

PROOF. There is a function h which does not vanish on $b\Omega$ such that $r' = hr$. On $b\Omega$, $\partial r' = h\partial r$ and $\partial \bar{\partial} r' = h\partial \bar{\partial} r + \partial h \wedge \bar{\partial} r + \partial r \wedge \bar{\partial} h$. Hence, on $b\Omega$,

$$\partial r' \wedge (\partial \bar{\partial} r')^{n-1} = h^n \partial r \wedge (\partial \bar{\partial} r)^{n-1} + \nu \wedge \partial r \wedge \bar{\partial} r$$

where ν is a $2n-3$ form. Since $\partial r \wedge \bar{\partial} r = dr \wedge \bar{\partial} r$, $i^*(\nu \wedge \partial r \wedge \bar{\partial} r) = 0$. Thus $i^*(\partial r' \wedge (\partial \bar{\partial} r')^{n-1}) = h^n i^*(\partial r \wedge (\partial \bar{\partial} r)^{n-1})$. Since $\sum r'_i(\zeta)(\zeta_i - z_i) = h(\zeta) \sum r_i(\zeta)(\zeta_i - z_i)$ for $\zeta \in b\Omega$, it follows from (2) that $i_*^* K_r(\zeta, z) = i_*^* K_{r'}(\zeta, z)$.

REMARK. It follows immediately from Corollary 2 and Theorem 3 that if one writes Aizenberg's kernel as $(H(\zeta, z)/(\Phi(\zeta, z))^n)d\sigma(\zeta)$, where $H(\zeta, z) = cJ(r)(\zeta)/|dr(\zeta)|$ and $\Phi(\zeta, z) = \sum r_i(\zeta)(\zeta_i - z_i)$, then $H(\zeta, z) \neq 0$ for $\zeta \in b\Omega$. In strictly pseudoconvex domains, Henkin's kernels can also be written in the form $(H(\zeta, z)/(\Phi(\zeta, z))^n)d\sigma(\zeta)$ for certain functions H and Φ . In this case, it is still true that $H(\zeta, \zeta) \neq 0$ for $\zeta \in b\Omega$ [1]. However, the proof is quite difficult because H is not given by an explicit formula.

BIBLIOGRAPHY

1. P. Ahern and R. Schneider, *The boundary behavior of Henkin's kernel*, Pacific J. Math. **66** (1976), 9–14.
2. L. A. Aizenberg, *Integral representations of functions which are holomorphic in convex regions of C^n space*, Dokl. Akad. Nauk SSSR **151** (1963), 1247–1249; English. transl., Soviet Math. Dokl. **4** (1963), 1149–1152.
3. C. L. Fefferman, *Monge-Ampère equations, the Bergman kernel and geometry of pseudoconvex domains*, Ann. of Math. **103** (1976), 395–416.
4. F. Norguet, *Introduction aux fonctions de plusieurs variables complexes: représentations intégrales*, Fonctions de Plusieurs Variables Complexes, 1–97, Lecture Notes in Math., no. 409, Springer-Verlag, Berlin and New York, 1974.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027