A THEOREM ON COMPACT LOCALLY CONFORMAL KÄHLER MANIFOLDS

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ABSTRACT. We prove that a compact locally conformal Kähler manifold which satisfies either: (1) it has nonpositive conformal invariant μ [2] and its local conformal Kähler metrics have nonnegative scalar curvature or (2) its local conformal Kähler (l.c.K.) metrics have a positive or negative definite Ricci form is a Kähler manifold. We conjecture that every compact l.c.K. manifold which satisfies all the topological restrictions of a Kähler manifold admits some Kähler metric.

A locally conformal Kähler (l.c.K.) manifold is a Hermitian manifold whose metric is conformal to a Kähler metric in some neighbourhood of every point. An interesting example of such manifolds is offered by the Hopf manifolds [5], which are compact and have no Kähler metric at all. Thereby, it follows that the result stated in [1] that every compact l.c.K. manifold is Kähler is not true in the general case, and we have the problem of characterising those compact l.c.K. manifolds which also admit Kähler metrics.

In this note, we give some sufficient conditions for a compact l.c.K. manifold to be Kähler and we formulate a general conjecture.

1. Let M^{2n} ($n > 1$) be a Hermitian manifold with metric g , complex structure J and fundamental form Ω . M is l.c.K. if every $x \in M$ has an open neighbourhood U with a differentiable function $\sigma: U \rightarrow \mathbb{R}$ such that

$$g'_U = e^{-\sigma}g|_U \quad (1)$$

is a Kähler metric on U . If we can take $U = M$, the manifold is *globally conformal Kähler* (g.c.K.).

Consider again the Hermitian manifold M . Then we can define the *Lee form* of M by

$$\omega = (1/(n-1))i(\Omega)d\Omega \quad (2)$$

and the corresponding *Lee vector field* $B = \sharp\omega$, where $\sharp$ denotes the operation of lifting indices by g .

Another interesting object is the linear connection given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B, \quad (3)$$

where ∇ is the Levi-Civita connection of g . $\tilde{\nabla}$ is called the *Weyl connection* [5] and for every notion defined by $\tilde{\nabla}$ we shall add the label *Weyl*. For

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example, we have the *Weyl curvature* $\tilde{R}$ of M , the *Weyl-Ricci curvature* $\tilde{\rho}$, the *Weyl scalar curvature* $\tilde{k}$, etc.

Note also the following property of $\tilde{\nabla}$ [5]:

$$(\tilde{\nabla}_X g)(Y, Z) = \omega(X)g(Y, Z), \quad (4)$$

for every vector field X, Y, Z on M .

Using the objects defined above, several characterizations of the l.c.K. manifolds are available:

THEOREM 1. *A Hermitian manifold (M, J, g) is l.c.K. iff it satisfies one of the following equivalent conditions:*

- (a) $d\Omega = \omega \wedge \Omega$;
- (b) $\tilde{\nabla}_X J = 0$;
- (c) $\tilde{\nabla}_X \Omega = \omega(X)\Omega$;
- (d) $\tilde{\nabla}_X$ preserves the type of the vector fields.

PROOF. We proved (a) and (b) in [5] and there we also saw that, for an l.c.K. manifold, ω is closed and locally $\omega = d\sigma$ with σ given in (1). To get (c), we obtain first by a direct computation

$$(\tilde{\nabla}_X \Omega)(Y, Z) = (\tilde{\nabla}_X g)(Y, JZ) + g(Y, (\tilde{\nabla}_X J)Z). \quad (5)$$

Formulas (4) and (5) lead easily to the equivalence of (b) and (c).

Finally, condition (d) means that, when $\tilde{\nabla}$ is extended by linearity to the complexification of the tangent bundle, $\tilde{\nabla}_X Z$ is of type $(1, 0)$ or $(0, 1)$ whenever Z has the same type, i.e. we have $JZ = \pm \sqrt{-1} Z$. But, the formula

$$(\tilde{\nabla}_X J)(Z) = \tilde{\nabla}_X(JZ) - J(\tilde{\nabla}_X Z) \quad (6)$$

and the fact that every Z is a sum of a vector of type $(1, 0)$ and one of type $(0, 1)$, yield easily the equivalence of (d) and (b).

To prove our main theorem we also need

THEOREM 2. *If M is an l.c.K. manifold, $\tilde{\nabla}$ is the Levi-Civita connection of the local Kähler metrics g'_U , $\tilde{\rho}$ is the Ricci curvature of g'_U and $e^\sigma \tilde{k}$ is the scalar curvature of g'_U .*

PROOF. We proved the first assertion in [5] and the second assertion is a corollary of the first. As for the last assertion, the scalar curvature of g' is

$$\text{Tr}(g'^{-1}\tilde{\rho}) = e^\sigma \text{Tr}(g^{-1}\tilde{\rho}) = e^\sigma \tilde{k}, \quad \text{q.e.d.}$$

2. In order to formulate the main theorem of this note, let us recall that every compact Riemannian manifold has a conformal invariant μ , the *Yamabe invariant*, which contains the essential information about the behaviour of the scalar curvature by conformal changes of the metric. We refer the reader to [2] for the definition and the properties of this invariant μ , and we shall prove here:

THEOREM 3. *If (M, J, g) is a compact l.c.K. manifold with the Yamabe invariant $\mu < 0$ and whose local Kähler metrics g'_U have nonnegative scalar curvature, then (M, J, g) is a globally conformal Kähler manifold.*

PROOF. Let us prove first that if g has a nonpositive scalar curvature and if M satisfies the above hypotheses, then g is a Kähler metric.

From formula (1) we have $g = e^a g'_U$, and, if we denote by k the scalar curvature of g , we get, by the classical relation between the scalar curvature of two conformally related Riemannian metrics [3] and using Theorem 2 above,

$$e^a k = e^a \tilde{k} - (2n-1)g'^{ij}\tilde{\nabla}_j \omega_i - \frac{1}{2}(2n-1)(n-1)g'^{ij}\omega_i \omega_j \quad (i, j = 1, \dots, 2n). \quad (7)$$

Now, let us notice the following result

LEMMA 1. *Let g_1 and g_2 be conformally related Riemannian metrics ($g_2 = fg_1$) on an m -dimensional orientable manifold. Then the corresponding codifferentials are related by*

$$\delta_2 \alpha = f^{-1} \delta_1 \alpha - \left(\frac{m}{2} - \deg \alpha \right) f^{-2} i_1(df) \alpha, \quad (8)$$

where i_1 denotes interior product for the metric g_1 .

The proof of this lemma follows from the known relation $\delta = (-1)^{p+1} d^*$ [3] and it is left to the reader.

Taking $g' = g_1$ and $g = g_2$ in (8) we get

$$\delta \omega = -g^{ij}\tilde{\nabla}_j \omega_i - (n-1)|\omega|^2, \quad (9)$$

and formula (7) becomes

$$k = \tilde{k} + (2n-1)\delta \omega + \frac{1}{2}(n-1)(2n-1)|\omega|^2, \quad (10)$$

which is an interesting formula by itself.

If M is compact, we get by integrating (10):

$$(\omega, \omega) = \frac{2}{(n-1)(2n-1)} \int_M (k - \tilde{k}) dv, \quad (11)$$

where dv denotes the volume element of g .

The integral formula (11) shows that $k - \tilde{k} < 0$ implies $\omega = 0$. This happens in our case because we have $k < 0$ and $\tilde{k} > 0$.

Hence, in this case, we have, as stated, that g is a Kähler metric.

Now, we come back to the initial conditions of Theorem 3. From a known theorem of Yamabe [2], it follows by $\mu < 0$ that M has a metric g^* conformal to g and having constant nonpositive scalar curvature.

It is obvious that (M, J, g^*) is again l.c.K. with respect to the same system of local Kähler metrics g'_U . Hence, by the above proof, g^* is Kähler, which ends the proof of Theorem 3.

It is also worth noting the following consequence of formula (10):

THEOREM 4. *If two of the following three properties:*

- (1) $k - \tilde{k} = \text{const}$;
- (2) *the Lee form ω is harmonic*;
- (3) *the Lee form ω has constant length; hold on an l.c.K. manifold M , then the third property also holds.*

PROOF. Since ω is closed, we get by differentiating (10):

$$d(\tilde{k} - k) + (2n - 1)\Delta\omega + \frac{1}{2}(n - 1)(2n - 1)d(|\omega|^2) = 0, \quad (12)$$

which obviously proves the theorem.

REMARK. As a matter of fact, Theorems 3 and 4 hold in the more general case of almost l.c.K. manifolds [5] since the above proofs are valid in this case as well. Of course in this case the conclusion of Theorem 3 will be that (M, J, g) is almost Kähler.

3. Theorem 3 yields a case when a compact l.c.K. manifold has a Kähler metric which is globally conformal to the initial metric.¹ We should, however, be interested in the more general question of finding conditions for an l.c.K. manifold to admit some Kähler metric, generally, unrelated to the given one.

For instance, we can easily prove

THEOREM 5. *If for the l.c.K. manifold M the local Kähler metrics g'_U have a (positive- or negative-) definite Ricci tensor (particularly if they are Einstein metrics with nonvanishing scalar curvature), then M admits a Kähler metric. If, moreover, M is compact, then M is a projective algebraic nonsingular variety.*

PROOF. In view of Theorem 2, the Ricci tensor of the hypothesis is just the Weyl-Ricci tensor $\tilde{\rho}$ and because the metrics g'_U are Kähler, $\tilde{\rho}$ is Hermitian with respect to J and it has the associated quadratic exterior form

$$\tilde{s}(X, Y) = \tilde{\rho}(X, JY), \quad (13)$$

called the Weyl-Ricci form of M .

Using again the Kählerian character of g'_U we can write [4]

$$\tilde{s} = \frac{1}{2}d'd''\ln \det g'_U, \quad (14)$$

where $d = d' + d''$ is the decomposition of the exterior differential d into terms of type $(1, 0)$ and $(0, 1)$, respectively.

In view of (1), (14) yields

$$\tilde{s} = -\sqrt{-1} s - nd'\omega, \quad (15)$$

where s is the Ricci form of the Hermitian metric g [4].

Since ω is closed, $d'\omega$ is closed as well and we get from (15) that the first real Chern class $C_1(M)$ is represented by the form $-(1/2\pi\sqrt{-1})\tilde{s}$.

¹This also happens in the following cases: (1) the first Betti number of the manifold vanishes [5], (2) $\delta(\Omega^h) = 0$ for some $h \neq 0, n$, (3) the manifold is complex analytically reducible (to be proved elsewhere).

Now, we see that, under the hypotheses of Theorem 5, $\pm(1/2\pi)\tilde{\rho}$ is positive-definite Hermitian and has an associated closed fundamental form, hence it is a Kähler metric on M . Moreover, since $C_1(M)$ comes from an integer cohomology class, we see that when M is compact it is a compact Hodge manifold. Q.E.D.

It is important to find conditions for an l.c.K. manifold to admit some Kähler metric because then all its topological and complex-analytic properties are as in the Kählerian case.

In order to suggest further studies of this subject, we shall state here a conjecture which is the most optimistic one but, until now, has not been well supported:

Conjecture. Every compact l.c.K. manifold which satisfies all the topological restrictions of the compact Kähler manifolds admits some Kähler metric.

As shown by the example of the Hopf manifolds [5] the imposed topological condition is essential.

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