MOMENT CONDITIONS FOR THE EXISTENCE AND NONEXISTENCE OF OPTIMAL STOPPING RULES FOR S_n/n^1

Y. S. CHOW AND JACK CUZICK

ABSTRACT. Sufficient conditions of moment type are given for the existence of optimal stopping rules for S_n/n . These results make use of a generalization of a result of Dini. In general it is shown that necessary conditions of moment type do not exist. In particular, we exhibit a symmetric random variable with $EX^+(\log^+ X)^{1+\varepsilon} = \infty$ for which an optimal rule exists. However under regularity conditions, we show that the sufficient condition given is very nearly necessary. Some results on exponentials of regularly varying functions are of independent interest.

1. Introduction. Let $X, X_1, X_2, \dots$ be i.i.d. random variables with mean zero, distribution function F , and partial sums $S_n = \sum_{i=1}^n X_i$. Assume that $P(X > x) > 0$ for some $x > 0$ and that $EX^+ \log^+ X < \infty$ where $x^+ = \max(0, x)$ and $\log^+ x = \max(1, \log x)$. When X is symmetric, Klass [1973, Theorems 8 and 9] has shown that a necessary and sufficient condition for the existence of an optimal stopping rule for S_n/n is that the integral

$$I = \int_1^\infty \left[\int_{x^+}^\infty y \log\left(\frac{y}{x}\right) dF(y) \right]^{-1} x dF(x) \quad (1)$$

should diverge. When X is not symmetric $I = \infty$ remains a sufficient condition. In this note we give some simple sufficient conditions for $I = \infty$ and for $I < \infty$. We begin in §2 by transforming (1) into a simpler expression. From this we then show that the finiteness of

$$EX^+ \exp(\log^+ X / \log^+ \log^+ X)^{1/2} \quad (2)$$

implies $I = \infty$. This improves a result of Chow and Lan [1975] who showed that a stronger moment condition implied the existence of an optimal stopping rule. The proof requires use of a generalization of a result of Dini which is of independent interest. In the final section we study the case when (2) is infinite. Here the situation is more complicated. In particular given any $\varepsilon > 0$ we shall exhibit a symmetric random variable X with $EX^+(\log^+ X)^{1+\varepsilon} = \infty$ for which $I = \infty$. Our example contains large gaps in which F places no mass. When some mild regularity conditions are placed on the tails of F , we obtain a simple condition slightly weaker than the finiteness of (2) which is necessary and sufficient for $I = \infty$.

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2. A sufficient condition. After the change of variables

$$G(\log x) = \int_0^x t \, dF(t)$$

we see from (1) that

$$I = \int_0^\infty \left(\int_{x+}^\infty (y - x) \, dG(y) \right)^{-1} dG(x). \quad (3)$$

We shall make use of this form for I below. First we need the following lemma, which is a generalization of a result of Dini (cf. Knopp [1948, p. 293]) who considered the case $\phi(x) \equiv 1$.

LEMMA 1. Let $F(x)$ be a finite measure on $[0, \infty)$ and assume that $H(x) = F(\infty) - F(x) > 0$ for all x . Also assume that $\phi(x)$ is a positive increasing differentiable function on $[1, \infty)$ tending to infinity. If

$$\limsup_{x \rightarrow \infty} \left(-\frac{\log H(x)}{\phi(x)} \right) > 0, \quad (4)$$

then, for all $K > 1$,

$$\int_K^\infty \frac{dF(x)}{\phi(x)H(x)} = \infty. \quad (5)$$

Further, if

$$\liminf_{x \rightarrow \infty} \left(-\frac{\log H(x)}{\phi(x)} \right) > 0, \quad (6)$$

then

$$\int_K^\infty \frac{Df(x)}{\phi(x) \log^+(\phi(x))H(x)} = \infty. \quad (7)$$

REMARKS. (i) The function $\log^+ \phi(x)$ in (7) can be replaced by

$$\log_k^+ \phi(x) \equiv \prod_{j=1}^k \log_j^+ \phi(x),$$

where $\log_2^+ x = \log^+ \log^+ x$, etc. for any $1 \leq k \leq \infty$.

(ii) The conclusions of the lemma continue to hold if x is restricted to continuity points of H in (6).

PROOF. We first establish (7). Let $\phi(x) > e$ for $x > K$. When x and y are continuity points of F , $y > x > K$, we use integration by parts to get

$$\begin{aligned} \int_x^y \frac{dF(t)}{\phi(t) \log \phi(t) H(t)} &= - \int_x^y \frac{dH(t)}{\phi(t) \log \phi(t) H(t)} \\ &= - \frac{\log H(t)}{\phi(t) \log \phi(t)} \Big|_x^y - \int_x^y \frac{\log H(t)}{\phi(t)} \frac{\phi'(t)(\log \phi(t) + 1)}{\phi(t)(\log \phi(t))^2} dt \\ &= \frac{h(y)}{\log \phi(y)} - \frac{h(x)}{\log \phi(x)} + \int_x^y h(t) \frac{\phi'(t)(\log \phi(t) + 1)}{\phi(t)(\log \phi(t))^2} dt \end{aligned} \quad (8)$$

where $h(t) = -\log H(t)/\phi(t)$.

By (6) we can choose K so that $h(t) > \varepsilon > 0$ for $t > K$. The first term in (8) remains bounded below as $y \rightarrow \infty$ and the second term is greater than

$$\varepsilon \int_x^y \frac{\phi'(t) dt}{\phi(t) \log \phi(t)} = \varepsilon \log_2 \phi(t) \Big|_x^y \rightarrow \infty.$$

To prove (5) assume (4) holds and (6) fails. Integrating by parts gives

$$\int_x^y \frac{dF(t)}{\phi(t)H(t)} = h(y) - h(x) + \int_x^y h(t) \frac{\phi'(t)}{\phi(t)} dt, \quad x, y \in \mathcal{C}(F). \quad (9)$$

Now choose K so that $H(x) < 1$ for $x > K$. Then the last term in (9) is positive. Since (6) fails and (4) holds there exists $\varepsilon > 0$ such that for any given $x_1 > K$, we can find $y > x > x_1$ with $x, y \in \mathcal{C}(F)$ such that $h(y) - h(x) > \varepsilon$. Since this is true for any such x_1 the result follows.

THEOREM 1. *Let F be a finite nontrivial measure on $[1, \infty)$. If*

$$\int_{e^*}^{\infty} x \exp \left\{ \frac{\log x}{\log_2 x} \right\}^{1/2} dF(x) < \infty, \quad (10)$$

then

$$I = \int_1^{\infty} \left[\int_{x+}^{\infty} y \log \left(\frac{y}{x} \right) dF(y) \right]^{-1} x dF(x) = \infty.$$

When $M = \inf\{x: F(\infty) - F(x) = 0\} < \infty$, the upper limits should be interpreted as $M +$.

REMARK. By use of the remark following Lemma 1, we can replace $\log_2 x$ in (10) by $\log_k^*(\log x)$ for any $1 < k < \infty$.

PROOF. When $M < \infty$, the theorem is easily proven. When $M = \infty$, equation (10) implies that

$$\tilde{G}(x) = \int_e^x \exp \left\{ \frac{1}{2} \left(\frac{y}{\log y} \right)^{1/2} \right\} dG(y) \quad (11)$$

is a bounded measure. Then

$$\begin{aligned} G^*(x) &= \tilde{G}(\infty) - \tilde{G}(x) = \int_x^{\infty} \exp \left\{ \frac{1}{2} \left(\frac{y}{\log y} \right)^{1/2} \right\} dG(y) \\ &\leq \exp \left\{ -\frac{1}{2} \left(\frac{x}{\log x} \right)^{1/2} \right\} \int_x^{\infty} \exp \left\{ \left(\frac{y}{\log y} \right)^{1/2} \right\} dG(y). \end{aligned} \quad (12)$$

For x large enough, by (10) the last factor in (12) is less than unity so that

$$\liminf_{x \rightarrow \infty} \left\{ -\log G^*(x) \left(\frac{x}{\log x} \right)^{-1/2} \right\} > \frac{1}{2}.$$

Thus (6) is satisfied with $\phi(x) = (x/\log x)^{1/2}$ so that Lemma 1 gives

$$\begin{aligned} \int_K^{\infty} \left[(x \log x)^{1/2} \int_x^{\infty} \exp \left\{ \frac{1}{2} \left(\frac{y}{\log y} \right)^{1/2} \right\} dG(y) \right]^{-1} \\ \cdot \exp \left\{ \frac{1}{2} \left(\frac{x}{\log x} \right)^{1/2} \right\} dG(x) = \infty \end{aligned} \quad (13)$$

for any $K > 1$. To show that (13) implies $I = \infty$, it is enough to show that

$$g(y) = \frac{(y-x)}{(x \log x)^{1/2}} \exp \left\{ \frac{1}{2} \left[\left(\frac{x}{\log x} \right)^{1/2} - \left(\frac{y}{\log y} \right)^{1/2} \right] \right\} \quad (14)$$

is bounded for all $y > x > K$ uniformly in x . Setting $g'(y) = 0$ gives

$$(y-x) = \frac{4(y \log y)^{1/2}}{1 - (\log y)^{-1}}. \quad (15)$$

This equation has only one solution and it corresponds to the maximum for g . When K is large enough it can easily be seen that $y < 2x$ and substituting this into the left-hand side of (15) gives

$$y-x < 8\sqrt{2} (x \log 2x)^{1/2} < 16(x \log x)^{1/2}.$$

Substituting into (14), we have that $g(y) < 16$ for $y > x > K$.

3. A partial converse. The hope that the divergence of some moment of X would imply $I < \infty$ is dashed by the following example. Given $\varepsilon > 0$, we shall exhibit a random variable with

$$EX^+ \log^+ X < \infty,$$

$$EX^+ (\log^+ X)^{1+\varepsilon} = \infty, \quad (16)$$

for which $I = \infty$. Let $N_k = 2^{2^k}$ and define X to be symmetric in such a way that the measure G in (3) places mass $CN_k^{-(1+\varepsilon)}$ at N_k with C chosen so that $F(\infty) = 1$. Then (16) is easily verified. It remains to show $I = \infty$. We have that

$$\int_{N_k}^{\infty} (y - N_k) dG(y) < C \sum_{j=k+1}^{\infty} N_j^{-\varepsilon} < C(1 - 2^{-\varepsilon})^{-1} N_{k+1}^{-\varepsilon}.$$

Thus from (3) we see that

$$I > (1 - 2^{-\varepsilon}) \sum_{k=1}^{\infty} \frac{N_k^{-(1+\varepsilon)}}{N_{k+1}^{-\varepsilon}} = (1 - 2^{-\varepsilon}) \sum_{k=1}^{\infty} 2^{e^{4^k} - (1+\varepsilon)2^{2^k}} = \infty.$$

This example has very large gaps in which F places no mass. If we rule out such "lacunary" random variables by prescribing some regularity on the upper tail of F , then a converse to Theorem 1 can be given. We begin with a further manipulation of (3). Let $G_1(x) = G(\infty) - G(x)$ so that

$$\int_x^{\infty} (y-x) dG(y) = \int_x^{\infty} G_1(y) dy.$$

Another integration by parts, now on the outer integral in (3), shows that

$$\begin{aligned} I &= - \left(\int_x^{\infty} G_1(y) dy \right)^{-1} G_1(x) \Big|_0^{\infty} \\ &\quad + \int_0^{\infty} \left(\left[\int_x^{\infty} G_1(y) dy \right]^{-1} G_1(x) \right)^2 dx. \end{aligned} \quad (17)$$

Our basic regularity condition is that $-\log G_1(x)$ is regularly varying at infinity.

DEFINITION. A function $f(x)$ is regularly varying at infinity with exponent α , if $f(x) > 0$ for $x > x_0$ and

$$\lim_{x \rightarrow \infty} \frac{f(sx)}{f(x)} = s^\alpha, \quad s > 0.$$

When $\alpha = 0$, $f(x)$ is said to be slowly varying.

We begin by recording the following standard fact.

LEMMA 2. If $f(x)$ is regularly varying at infinity with exponent α , then for all $\varepsilon > 0$, there exists x_0 such that

$$s^{\alpha-\varepsilon} < \frac{f(sx)}{f(x)} < s^{\alpha+\varepsilon}, \quad s > 1, x > x_0.$$

PROOF (Seneta [1976, p. 18]). The following lemma describes the relationship between exponentials of regularly varying functions and their integrals. Although stated for iterated and fractional integrals, it will only be used below for the simple integrals in (17).

LEMMA 3. Let $f(x)$ be regularly varying at infinity with exponent $\alpha > 0$. Assume $\beta > -1$ and for $x > x_0$ define

$$g(x) = \frac{1}{e^{-f(x)}} \int_x^\infty (t-x)^\beta e^{-f(t)} dt = \int_x^\infty (t-x)^\beta e^{-[f(t)-f(x)]} dt.$$

Then

$$g(x) \left(\frac{f(x)}{x} \right)^{1+\beta} \xrightarrow{x \rightarrow \infty} \alpha^{-(1+\beta)} \Gamma(1+\beta)$$

so that $g(x)$ is regularly varying with exponent $(1+\beta)(1-\alpha)$.

PROOF. Let $y = ((t-x)/x)f(x)$. Then

$$\gamma(x) \equiv g(x) \left(\frac{f(x)}{x} \right)^{1+\beta} = \int_0^\infty h_x(y) dy$$

where

$$h_x(y) = y^\beta \exp \left\{ f(x) - f \left(x + \frac{xy}{f(x)} \right) \right\}.$$

By Lemma 2,

$$h_x(y) \xrightarrow{x \rightarrow \infty} y^\beta e^{-\alpha y}.$$

Hence if $h_x(y) < H(y)$ with $\int_0^\infty H(y) dy < \infty$, it follows by dominated convergence that

$$\gamma(x) \xrightarrow{x \rightarrow \infty} \alpha^{-(1+\beta)} \Gamma(1+\beta).$$

To establish the existence of such H , first let $\alpha > 1$. Choose $\varepsilon > 0$ so that $\alpha - \varepsilon > 1$. By Lemma 2, for $x > x_0$,

$$f(x) - f\left(x + \frac{xy}{f(x)}\right) < f(x) \left\{ 1 - \left(1 + \frac{y}{f(x)}\right)^{\alpha-\epsilon} \right\} < -(\alpha - \epsilon)y, \quad (18)$$

the last inequality following from $(1+z)^{\alpha-\epsilon} - 1 > (\alpha - \epsilon)z$, for $z > 0$. Hence

$$h_x(y) < y^\beta e^{-(\alpha-\epsilon)y} \equiv H(y)$$

and $H(y)$ is integrable. Now let $0 < \alpha < 1$. For $\epsilon > 0$ and $\alpha - \epsilon > 0$, again by Lemma 2 the first inequality in (18) holds. Choose $x_0 > 0$ so that $f(x) > 1$ for $x > x_0$. When $0 < z < 1$,

$$(1+z)^{\alpha-\epsilon} - 1 > (\alpha - \epsilon)z(1+z)^{\alpha-1-\epsilon} > (\alpha - \epsilon)2^{\alpha-1-\epsilon}z,$$

so that

$$f(x) \left\{ 1 - \left(1 + \frac{y}{f(x)}\right)^{\alpha-\epsilon} \right\} < -(\alpha - \epsilon)2^{\alpha-1-\epsilon}y, \quad y < f(x). \quad (19)$$

When $z > 1$, $(1+z)^{\alpha-\epsilon} - 1 > (2^{\alpha-\epsilon} - 1)z^{\alpha-\epsilon}$ so that

$$\begin{aligned} f(x) \left\{ 1 - \left(1 + \frac{y}{f(x)}\right)^{\alpha-\epsilon} \right\} &< -(2^{\alpha-\epsilon} - 1)y^{\alpha-\epsilon}f(x)^{1-(\alpha-\epsilon)} \\ &< -(2^{\alpha-\epsilon} - 1)y^{\alpha-\epsilon}, \quad y > f(x). \end{aligned} \quad (20)$$

Combining (19) and (20) gives

$$h_x(y) < y^\beta [\exp\{-(\alpha - \epsilon)2^{\alpha-1-\epsilon}y\} + \exp\{-(2^{\alpha-\epsilon} - 1)y^{\alpha-\epsilon}\}] \equiv H(y)$$

and again H is integrable.

We can now use Lemma 3 to establish a converse to Theorem 1.

THEOREM 2. Assume $EX^+ \log^+ X^+ < \infty$ (or equivalently that $\int x dG(x) < \infty$) and that $G_1(x) = G(\infty) - G(x)$ is such that $-\log G_1(x)$ is regularly varying at infinity with exponent α (necessarily $\alpha > 0$).

If $\alpha > \frac{1}{2}$ then $I = \infty$.

If $\alpha < \frac{1}{2}$ then $I < \infty$.

If $\alpha = \frac{1}{2}$, then $-x^{-1/2} \log G_1(x) \equiv L(x)$ is slowly varying and

$$I = \infty \Leftrightarrow \int_{x_0}^{\infty} \frac{L^2(x)}{x} dx = \infty.$$

PROOF. When $\alpha > 0$, the proof obtains by applying Lemma 3 to (17) with $f(x) = -\log G_1(x)$ and $\beta = 0$. When $\alpha = 0$, $f(x)$ is slowly varying. Taking $\beta = 0$ we show that the $g(x)$ of Lemma 3 satisfies

$$g(x) > x^{1-\epsilon} \quad \text{for all } \epsilon > 0, \quad x > x_0.$$

After the change of variables $t = sx$,

$$g(x) = x \int_1^{\infty} \exp \left\{ \left[1 - \frac{f(sx)}{f(x)} \right] f(x) \right\} ds$$

which for any $1 > \delta > 0$ and $x > x_0 = x_0(\delta)$, by Lemma 2 is greater than

$$\begin{aligned} x \int_1^\infty \exp\{(1-s^\delta)f(x)\} ds &= x \int_0^\infty \exp\{[1-(1+s)^\delta]f(x)\} ds \\ &> x \int_0^\infty \exp\{-s^\delta f(x)\} ds \end{aligned}$$

since $(1+z)^\delta < 1+z^\delta$ for $0 < \delta < 1$. Making the further change of variables $s' = s(f(x))^{1/\delta}$ this equals

$$\frac{x}{(f(x))^{1/\delta}} \int_0^\infty \exp(-s^\delta) ds \sim \frac{x}{(f(x))^{1/\delta}}.$$

Since $f(x)$ is slowly varying, $f(x) < x^{\epsilon\delta}$ for large x implying that for some x_0

$$g(x) > x^{1-\epsilon}, \quad x > x_0.$$

Since $g(0) > 0$, we see that (17) is finite in this case also.

Theorem 2 can be rephrased in terms of a moment condition as follows:

COROLLARY 1. *Under the conditions of Theorem 2, if*

$$EX^+ \exp(\log^+ X)^\delta = \infty \quad \text{for } \delta < \frac{1}{2} \quad (21)$$

then $I < \infty$.

PROOF. First note that

$$\int_1^\infty xf(x) dF(x) = - \int_1^\infty f(e^x) dG_1(x). \quad (22)$$

When (21) holds and $f(x) = \exp(\log^+ x)^\delta$ we have

$$- \int_1^\infty \exp(x^\delta) dG_1(x) = \infty$$

and since $G_1(x) = \exp(-x^\alpha L(x))$ with $L(x)$ slowly varying we must have $\alpha < \delta < \frac{1}{2}$, so that $I < \infty$.

REMARK. To get an idea of the appropriate moment condition in the boundary case $\alpha = \frac{1}{2}$, note that if we impose the further regularity condition

$$L(x) \equiv -x^{-1/2} \log G_1(x) = \exp \left\{ \int_{x_0}^x \frac{\varepsilon(u)}{u} du \right\}, \quad x > x_0 \quad (23)$$

for some $\varepsilon(u) \downarrow 0$, then

$$E(X^+ \exp\{\gamma(\log^+ X)^{1/2} L(\log^+ X)\}) = \begin{cases} +\infty, & \gamma > 1, \\ < \infty, & \gamma < 1. \end{cases} \quad (24)$$

To see this, when (23) holds check that

$$G_1'(x) = -G_1(x)x^{-1/2}L(x)\left(\frac{1}{2} + \varepsilon(x)\right)$$

and using (22), the left-hand side of (24) equals

$$\begin{aligned} & \int_0^\infty \exp\{\gamma x^{1/2} L(x)\} G_1(x) x^{-1/2} \left(\frac{1}{2} + \varepsilon(x)\right) dx \\ &= \int_0^\infty \exp\{-(1-\gamma)x^{1/2} L(x)\} x^{-1/2} \left(\frac{1}{2} + \varepsilon(x)\right) dx \end{aligned}$$

which is finite iff $\gamma < 1$. By Theorem 2, $I < \infty$ if $L(x) \sim (\log x)^{-(1/2+\varepsilon)}$, $\varepsilon > 0$ or

$$L(x) \sim [\log_k^*(x)(\log_{k+1}^+ x)^{1+\varepsilon}]^{-1/2}, \quad k \geq 1, \quad \varepsilon > 0.$$

However $I = \infty$ if $L(x) \sim (\log^+ x)^{-1/2}$ or even $L(x) \sim (\log_\infty^* x)^{-1/2}$. (This is always true by Theorem 1.) In combination with (24), this provides a good idea as to where the boundary for $I < \infty$ is, in terms of moments, for sufficiently regular distribution functions $F(x)$.

COROLLARY 2. *If $A \exp(-x^\delta L(x)) < G_1(x) < B \exp(-x^\delta L(x))$ for $0 < A < B < \infty$, $\delta < \frac{1}{2}$, and $L(x)$ slowly varying, then $I < \infty$.*

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DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027