THE HOMOLOGY 3-SPHERES WITH INVOLUTIONS

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ABSTRACT. Let Σ be a homology 3-sphere which supports an orientation reversing involution. Then $\mu(\Sigma) = 0$.

In this paper we study the homology 3-spheres which support orientation reversing involutions. Given such a homology 3-sphere, we show that it bounds an orientable, parallelizable 4-manifold of index zero. Hence it has Rohlin invariant 0.² (Compare this with Problem 4.4 of [3].)

Let Σ be a 3-manifold with $H_*(\Sigma) = H_*(S^3)$, and let g be an orientation reversing involution on Σ with fixed point set F . It follows easily from Smith theory, the Lefschetz fixed point theorem and the slice theorem that F is either a 2-sphere or just two points.

Suppose $F = S^2$. Then by [1, IV, 2.6, p. 179] we know that $\Sigma/Z_2 = \Sigma^*$ is an acyclic 3-manifold with boundary F , and it also follows that Σ is the boundary of $\Sigma^* \times I$. Hence Σ actually bounds an acyclic 4-manifold.

Suppose $F = \{\chi_0, \chi_1\}$. Let D_i be a closed g -invariant D^3 -neighborhood of χ_i ($i = 0, 1$) and $N = \Sigma - \text{Int}(D_0 \cup D_1)$. Notice that the orbit map $\Pi: N \rightarrow N^*$ is a 2-fold covering projection. ∂N^* has two connected components $\partial_+ N^*$ and $\partial_- N^*$. They are both diffeomorphic to RP^2 . Let $h_+: \partial_+ N^* \rightarrow RP^2$ be a diffeomorphism and $h: N^* \rightarrow RP^2$ be an extension of h_+ . (Such an extension always exists.) It can be easily seen that $h|_{\partial_- N^*}$ is homotopic to a diffeomorphism. Define $f: N^* \rightarrow RP^2 \times I$ canonically using h and this homotopy. Observe that $f_*: H_*(N^*) \rightarrow H_*(RP^2 \times I)$ is an isomorphism. Let $S^H(RP^2 \times I, \partial(RP^2 \times I))$ be the set of all concordance classes of maps (X, f) , where X is a 3-manifold with boundary, $f: X \rightarrow RP^2 \times I$ is a map which maps $\partial(X)$ diffeomorphically onto $\partial(RP^2 \times I)$ and induces an isomorphism in homology. (Note. (N^*, f) represents an element in $S^H(RP^2 \times I, (RP^2 \times I))$.) Following Browder [2] and Wall [6], let

$$\eta: S^H(RP^2 \times I, \partial(RP^2 \times I)) \rightarrow [RP^2 \times I, \partial(RP^2 \times I), G/0, *]$$

be the map defined by the normal invariant. Since

$$i^*: [RP^2 \times I, \partial(RP^2 \times I); G/0, *] \rightarrow [RP^1 \times I, \partial(RP^1 \times I); G/0, *]$$

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²This result was obtained first by J. Birman using different methods. Recently, D. Galewski and R. Stern gave yet another proof.

is an isomorphism, and

$$[RP^1 \times I, \partial(RP^1 \times I); G/0, *] \cong H^2(S(RP^1), Z_2) = Z_2,$$

$[RP^2 \times I, \partial(RP^2 \times I); G/0, *]$ has two elements. Of course one of them is represented by the identity map. It follows from [4] or [5, Theorem 2.2] that the nontrivial element in $[RP^2 \times I, \partial(RP^2 \times I); G/0, *]$ can be represented by a normal map (f, b) , where $f: RP^2 \times I \rightarrow RP^2 \times I$ is a map which induces an isomorphism in homology. Hence (N^*, f) is always normally cobordant rel boundary to a homology equivalence from $RP^2 \times I$ to itself. Let (W^*, F) be such a normal cobordism. Consider the normal map $F \times \text{id}: W^4 \times CP^2 \rightarrow (RP^2 \times I \times I) \times CP^2$. It gives an 8-dimensional surgery problem with surgery obstruction $\theta(F \times \text{id}) \in L_8^-(Z_2)$. Let

$$\tilde{F} \times \text{id}: \tilde{W}^4 \times CP^2 \rightarrow (S^2 \times I \times I) \times CP^2$$

be the lifting of $F \times \text{id}$ in the canonical double covers. $\tilde{F} \times \text{id}$ also gives us a surgery problem with a given homology equivalence on the boundary. Let $\theta(\tilde{F} \times \text{id}) \in L_8(1)$ be its surgery obstruction, and let $\rho: L^-(Z_2) \rightarrow L_8(1)$ be the transfer homomorphism. Then $\rho(\theta(F \times \text{id})) = \theta(\tilde{F} \times \text{id})$. But $L_8^-(Z_2) = Z_2$, $L_8(1) = Z$, [6, Theorem 13A.1], so $\rho: Z_2 \rightarrow Z$ must be the trivial map. Thus $\theta(\tilde{F} \times \text{id}) = 0$. However the 8-dimensional surgery obstruction is given by the index. So $\text{Index}(\tilde{W} \times CP^2) = 0$. $\text{Index}(\tilde{W}) = \text{Index}(\tilde{W} \times CP^2)$. Hence $\text{Index}(\tilde{W}) = 0$. Clearly $\tilde{W}$ is parallelizable, and

$$\partial \tilde{W} = N \cup S^2 \times I \cup S^2 \times I \cup S^2 \times I.$$

Let $U = \tilde{W} \cup D^3 \times I \cup D^3 \times I$, where these two copies of $D^3 \times I$ are attached to $\tilde{W}$ along the second and third copies of $S^2 \times I$ above in the obvious fashion. Now $\partial U = \Sigma \cup S^3$. Fill in D^4 along the S^3 in ∂U , we get a 4-manifold M with boundary Σ . Since M is constructed from $\tilde{W}$ by adding two 3-handles and one 4-handle, but still has a nonempty boundary, it is clear that M is still parallelizable and $\text{Index}(M) = 0$. In conclusion we have the following:

THEOREM. *Let M be a 3-manifold with $H_*(\Sigma) = H_*(S^3)$. Suppose Σ supports an C^∞ orientation reversing involution. Then Σ bounds an orientable, parallelizable 4-manifold with index zero. In particular, $\mu(\Sigma) = 0$.*

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