ON GROUPS WHICH ACT FREELY ON VECTOR BUNDLES OVER SPHERES

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ABSTRACT. Let $G \cong K \rtimes Q$ be the semidirect product of a finitely presented group K of type FP and a finite group Q of periodic cohomology. It is shown that G acts freely and properly discontinuously on a vector bundle over a sphere.

Let G be a finitely presented group of type VFP [5], and periodic Farrell cohomology. C.T.C. Wall has conjectured that G acts freely and properly discontinuously on the product of a sphere and a Euclidean space with, in general, noncompact quotient. The simplest form of this group theoretic behaviour occurs when G is a semidirect product $G \cong K \rtimes Q$ where K is a finitely presented group of type FP and Q is a finite group with periodic cohomology. In this note we show that such G act freely and properly discontinuously on a vector bundle over a sphere. In retrospect this seems a more natural formulation of the above conjecture, though we are unable to dispose of the conjecture in Wall's original formulation. We begin with

THEOREM 1. *Let $G \cong K \rtimes Q$ be the semidirect product of a finitely presented group K of type FL (as kernel) and a finite group Q with periodic cohomology. Then G acts freely and properly discontinuously on the total space of a smooth fibre bundle over a sphere with fibre a Euclidean space.*

PROOF. Write G as the middle term in an exact sequence thus: $\mathcal{E} = (K \twoheadrightarrow G \twoheadrightarrow Q)$ and let $B\mathcal{E}$ be the geometrical realisation of the associated minimal Kan fibration, $B\mathcal{E} = (BK \rightarrow BG \rightarrow BQ)$. $B\mathcal{E}$ is locally trivial in the category of compactly generated spaces [2]. By the main result of Swan's thesis [8], we may choose a finite simplicial complex Y_Q with fundamental group Q and universal covering $\tilde{Y}_Q$ homotopy equivalent to a k -sphere S^k , for some k . Let X_Q be an open regular neighbourhood of Y_Q in some high dimensional Euclidean space $\mathbb{R}^{m+k}$. Since S^k and $\tilde{Y}_Q$ have the same simple homotopy type, then $\tilde{X}_Q$ is also an open regular neighbourhood for S^k provided m is sufficiently large [9]. Hence we may assume that $\tilde{X}_Q$ is diffeomorphic to $\mathbb{R}^m \times S^k$.

By hypothesis on K we may choose a finite complex Y_K homotopy

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equivalent to BK . Let X_K be an open regular neighbourhood of Y_K in $\mathbb{R}^n$ for large enough n ($n > 2(m + k)$ at least). By taking a further product with $\mathbb{R}$ if necessary, we can suppose that $\tilde{X}_K$ is diffeomorphic to $\mathbb{R}^n$ as in [6], [4], so we assume this as well.

Let $c: X_Q \rightarrow BQ$ induce an isomorphism of fundamental groups, and put $\xi = c^*(B\mathbb{G}) = (BK \rightarrow E_\xi \rightarrow X_Q)$. ξ is locally trivial, since X_Q is locally compact, hence is a Hurewicz fibration. We now apply the open fibre smoothing theorem of Casson and Gottlieb (Theorem A of [1]) to deduce that ξ is fibre homotopy equivalent to a smooth fibre bundle $\eta = (X_K \rightarrow X_G \rightarrow X_Q)$ with fibre X_K and base X_Q . If $\hat{X}_G$ denotes the covering of X_G with $\pi_1(\hat{X}_G) = K$, then $\hat{X}_G$ is a smooth bundle over $\tilde{X}_Q = \mathbb{R}^m \times S^k$ with fibre X_K , hence the universal cover $\tilde{X}_G$ fibres over $\mathbb{R}^m \times S^k$ with fibre $\mathbb{R}^n \cong \tilde{X}_K$. By projecting onto S^k and using the contractibility of $\mathbb{R}^m$, we see that $\tilde{X}_G$ fibres smoothly over S^k with fibre $\mathbb{R}^{m+n}$. The result follows since G acts freely and properly discontinuously on $\tilde{X}_G$. Q.E.D.

To get from smooth $\mathbb{R}^N$ bundles to vector bundles, we can use the following piece of folklore, which is by no means the strongest possible statement.

PROPOSITION 2. *Let E be the total space of a smooth $\mathbb{R}^N$ bundle over a smooth closed manifold M^k . Suppose that (a) $k + N > 5$ and (b) $N > \max(k, 2)$. Then E is diffeomorphic to the total space of a vector bundle over M .*

PROOF. Since $\mathbb{R}^N$ is contractible, we may construct a section $i: M \rightarrow E$ for ξ , which, since $2 \dim M < \dim E$, we may suppose is an immersion. Hence $i(M)$ is a smooth submanifold of E . Let N_1, N_2 be smooth closed tubular neighbourhoods of $i(M)$ with $N_1 \subset \text{Int } N_2$. By uniqueness of tubular neighbourhoods, we see that $\text{Int } N_2 - \text{Int } N_1 \cong \partial N_1 \times [0, 1)$. Since $N > 3$, then $E - \text{Int } N_1$ is an open h -cobordism. Since $k + N > 5$, we may apply the open h -cobordism theorem of Stallings [7] to conclude that $E - \text{Int } N_1 \cong \partial N_1 \times [0, 1)$. Consequently, E is diffeomorphic to $\text{Int } N_2$, which is the total space of a normal vector bundle. Q.E.D.

Putting together Theorem 1 and Proposition 2 we obtain

THEOREM 3. *Let $G \cong K \rtimes Q$ be the semidirect product of a finitely presented group of type FP and a finite group Q with periodic cohomology. Then G acts freely and properly discontinuously on the total space of a vector bundle over a sphere.*

PROOF. If K is of type FL, the result follows from Theorem 1 and Proposition 2. It is easily checked that the construction of Theorem 1, provided $k > 1$ and $m > 1$, automatically satisfies the subsidiary conditions of Proposition 2. Now, it is unknown whether every finitely presented group of type FP is also of type FL. If K is not of type FL, then $K \times \mathbb{Z}$ is of type FL, by Gersten's product formula for Wall's finiteness obstruction [3]. Then

$G \times \mathbb{Z} \cong (K \times \mathbb{Z}) \rtimes Q$, and since $G \times \mathbb{Z}$ acts freely and properly discontinuously on a vector bundle over a sphere, so does G . Q.E.D.

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