A THEOREM ON PARTIALLY ORDERED SETS OF ORDER-PRESERVING MAPPINGS

DWIGHT DUFFUS AND RUDOLF WILLE¹

ABSTRACT. Let P be a partially ordered set and let P^P denote the set of all order-preserving mappings of P to P ordered by $f < g$ in P^P if $f(p) < g(p)$ for all $p \in P$. We prove that if P and Q are finite, connected partially ordered sets and $P^P \cong Q^Q$ then $P \cong Q$.

Is a partially ordered set determined by its order-preserving mappings? L. M. Gluskin [4] has shown that the set of order-preserving mappings of a partially ordered set P as a semigroup of transformations determines P up to duality. Let us consider the set P^P of order-preserving mappings of P as a partially ordered set: $f \leq g$ in P^P if $f(p) \leq g(p)$ for all $p \in P$. (If P is a finite lattice then P is determined by P^P [6].)

The purpose of this note is to establish the following.

THEOREM. *Let P and Q be finite, connected partially ordered sets. Then $P^P \cong Q^Q$ implies $P \cong Q$.*

In proving the theorem, we first change the problem from one concerning exponents X^Y of partially ordered sets to one about products XY (or $X \times Y$) of partially ordered sets. The change is accomplished with a description of the covering relation in an exponent.

Recall that for elements $a > b$ in a partially ordered set X , a covers b if $a > c > b$ implies $a = c$. Let $V(X)$ denote the set of elements of X with a unique lower cover in X . Let X^d denote the dual of X . The following result is due to D. Duffus and I. Rival [3].

Logarithmic property. *Let X and Y be finite partially ordered sets. If X has a least element, then $V(X^Y) \cong V(X) \times Y^d$.*

Let X be a finite partially ordered set. For $x \in X$ we let $[x] = \{y \in X \mid y \geq x\}$ and let $V(x) = V([x])$. We also let the length $l(X)$ of X be defined by

$$l(X) = \sup\{|C| - 1 \mid C \subseteq X, C \text{ is a chain}\}$$

and the depth $\delta(x)$ of x in X is given by

$$\delta(x) = \sup\{l(C) \mid C \subseteq X, C \text{ is a chain and } \inf(C) = x\}.$$

It is clear that x has maximum depth in X if and only if $\delta(x) = l(X)$.

Received by the editors February 16, 1978.

AMS (MOS) subject classifications (1970). Primary 06A10.

¹The work presented here was supported in part by the National Research Council of Canada.

Let P and Q be finite, connected partially ordered sets and let φ be an isomorphism of P^P onto Q^Q . Let $f \in P^P$ satisfy $\delta(f) = l(P^P)$. Since $l(X^Y) = l(X) \cdot |Y|$ for finite partially ordered sets X and Y [3], $\delta(f) = l(P) \cdot |P|$. Moreover,

$$\delta(f) \leq \sum_{a \in P} \delta(f(a));$$

therefore, $\delta(f(a)) = l(P)$ for all $a \in P$. As $f(P)$ is connected, we conclude that f is a constant mapping, say $f(P) = \{a\}$, where $\delta(a) = l(P)$. Similarly, if $\varphi(f) = g$ then g is an element of maximum depth in Q^Q and $g(Q) = \{b\}$, where $\delta(b) = l(Q)$. It is not difficult to see that

$$[a]^P \cong [f] \cong [g] \cong [b]^Q. \quad (1)$$

We apply the logarithmic property to (1) and obtain

$$V(a) \times P^d \cong V(b) \times Q^d. \quad (2)$$

Let us apply an extension of J. Hashimoto's common refinement theorem for product factorizations to (2) [5] (cf. [2]). We obtain partially ordered sets R, S, T , and U satisfying

$$V(a) \cong RS^d, \quad P \cong TU, \quad V(b) \cong RT^d, \quad Q \cong SU. \quad (3)$$

Let $a \in P$ correspond to $(t, u) \in TU$ and $b \in Q$ correspond $(s, u') \in SU$ under the isomorphisms in (3). Observe that $(x, y) \in XY$ satisfies $(x, y) \in V(XY)$ if and only if $x \in V(X)$ and y is a minimal element of Y or x is minimal in X and $y \in V(Y)$. Then $[a] \cong [t] \times [u]$ and $[b] \cong [s] \times [u']$; hence,

$$RS^d \cong V(a) \cong V(t) + V(u)$$

and

$$RT^d \cong V(b) \cong V(s) + V(u').$$

(We use $X + Y$ to denote the disjoint union of partially ordered sets X and Y [1].) S and T are connected partially ordered sets; therefore, $R \cong R_1 + R_2 \cong R_3 + R_4$ and $V(t) \cong R_1 S^d$, $V(s) \cong R_3 T^d$ [1]. It now follows that

$$|V(t)| = |R_1| |S| \geq |R_1| |V(s)| = |R_1| |R_3| |T|.$$

Since $t \notin V(t)$, $V(t)$ is a proper subset of T . Therefore, $R_1 = \emptyset$ or $R_3 = \emptyset$. Let us suppose that $R_1 = \emptyset$. Then $V(t) = \emptyset$ and, because t is a minimal element of the connected partially ordered set T , $|T| = 1$. Hence, from (3),

$$U^U \cong P^P \cong Q^Q \cong (S^{SU})(U^U)^S.$$

It follows that $|S| = 1$ and that $P \cong U \cong Q$. In the case that $R_3 = \emptyset$, a similar argument establishes that $P \cong Q$.

The proof of the theorem is now complete.

We observe that the same argument as used in the proof of the Theorem verifies a curious "cancellation" law.

COROLLARY. *Let P and Q be finite, connected partially ordered sets. Then $P^Q \cong Q^P$ implies $P \cong Q$.*

The corollary does not hold for all finite antichains n : $2^4 \cong 4^2!$

REFERENCES

1. G. Birkhoff, *Extended arithmetic*, Duke Math. J. **3** (1937), 311–316.
2. D. Duffus, *Toward a theory of finite partially ordered sets*, Ph.D. Thesis, University of Calgary, 1978.
3. D. Duffus and I. Rival, *A logarithmic property for exponents of partially ordered sets*, Canad. J. Math. **30** (1978), 797–807.
4. L. M. Gluskin, *Semigroups of isotone transformations*, Uspehi Mat. Nauk **16** (1961), 157–162. MR **24** #A1336.
5. J. Hashimoto, *On direct product decomposition of partially ordered sets*, Ann. of Math. (2) **54** (1951), 315–318. MR **13**, 201.
6. R. Wille, *Cancellation and refinement results for function lattices*, Houston J. Math. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALGARY, CALGARY, CANADA

DEPARTMENT OF MATHEMATICS, TECHNISCHE HOCHSCHULE DARMSTADT, DARMSTADT, FEDERAL REPUBLIC OF GERMANY