A CONDITION EQUIVALENT TO THE SOLVABILITY OF THE ENDOMORPHISM PROBLEM FOR FREE GROUPS

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ABSTRACT. To solve the endomorphism problem for a word W involving n generators of a free group F , it suffices to solve the problem assuming that the images of W involve no more than n generators.

1. Introduction. Let F be the free group freely generated by $x_1, x_2, \dots$. The endomorphism problem for the pair $(W, U) \in F \times F$ is to determine whether there is an endomorphism of F sending W to U . The endomorphism problem for a set P of pairs is said to be solvable if there is an effective algorithm which solves the endomorphism problem for every $(W, U) \in P$. The endomorphism problem for a word W is obtained by setting $P = \{W\} \times F$.

In this note we will show that it suffices to consider only those words U containing no more distinct generators than occur in W . Letting $g(W)$ be the cardinality of the set $\{x_i: x_i \text{ or } x_i^{-1} \text{ occurs in } W\}$, our condition can be stated more concisely as follows:

THEOREM. *The endomorphism problem is solvable for W if and only if it is solvable restricted to the set of pairs $\{(W, U): g(U) \leq g(W)\}$.*

The possibility of such a reduction was suggested to the author by M. J. Wicks (written communication). The reader is referred to [1] for a discussion of definitions and notation.

2. Proof of the theorem. The theorem follows from a combination of certain results in [1]. The necessity of the condition is clear, thus it remains to establish its sufficiency.

To solve the endomorphism problem for a pair (W, U) it is enough, by Whitehead's theorem [3], to solve the problem for some pair (W', U') where W' and U' are automorphic images of W and U , respectively. Therefore, as in [1], we can assume that W and U are reduced, irredundant, cyclic words. By Theorem 2.1 of [1], U is an endomorphic image of W if and only if U is a c -free image of some word in D_W . Letting $U_1, U_2, \dots, U_n$ be the distinct (cyclic) words having U as a c -free image, it follows that U is an endomorphic image of W if and only if some U_1 is in D_W .

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Our result is trivial if W is universal. If W is nonuniversal then so is every word in D_W , therefore we can omit any universal U_i from further consideration. Lemma 5.2 of [1] implies that for each U_i there is a connected word V_i , effectively calculable from U_i , which is an automorphic image of U_i with $\Delta(U_i) \geq \Delta(V_i)$. Clearly U is an endomorphic image of W if and only if some V_i is an endomorphic image of W . If $U_i \in D_W$, Proposition 3.8 of [1] implies that $\Delta(W) \geq \Delta(U_i)$. Thus $\Delta(W) \geq \Delta(U_i) \geq \Delta(V_i) = g(V_i) - 1$. It follows easily that $g(W) \geq g(V_i)$. Assuming then that the endomorphism problem is solvable for the set of pairs (W, V) with $g(V) \leq g(W)$ it follows that it is solvable for W .

REMARK. Note that we have proved the somewhat sharper result that the endomorphism problem is solvable for W if and only if it is solvable for the set of pairs (W, U) with U connected and $\Delta(U) \leq \Delta(W)$.

COROLLARY. Solving the endomorphism problem for a word W in F , with $g(W) = n$, is equivalent to solving the endomorphism problem for W in F_n ($= \langle x_1, x_2, \dots, x_n; \emptyset \rangle$).

REMARK. In [2] Paul Schupp solves the endomorphism problem for W any two-generator word in F_n ($n \geq 2$). This is equivalent to solving the endomorphism problem for F_2 (i.e., for every word in F_2).

The three-generator case is open. Here our result shows that, instead of studying the images of three-generator words in F , it is enough to solve the endomorphism problem for F_3 .

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