LOCALLY COMPACT GROUPS WHICH ARE AMENABLE AS DISCRETE GROUPS

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ABSTRACT. In the courses of their independent studies of the existence of invariant means which are not topologically invariant, E. Granirer and W. Rudin have considered several properties of a locally compact group G which are satisfied if G is amenable as a discrete group. By applying a result of J. Rosenblatt (together with ideas of Granirer and Rudin) we show some of these properties on G imply that G is amenable as a discrete group.

Let G be a locally compact group with a fixed left Haar measure λ . As usual, if G is compact then λ is assumed to be normalized, i.e., $\lambda(G) = 1$. Let $L^\infty(G)$ be the Banach algebra of bounded real-valued Borel functions on G with the λ -essential supremum norm $\|\cdot\|_\infty$. For $f \in L^\infty(G)$ and $x \in G$, $l_x f \in L^\infty(G)$ is defined by $(l_x f)(y) = f(xy)$, $y \in G$. For $\varphi \in L^\infty(G)^*$ and $x \in G$, $L_x \varphi \in L^\infty(G)^*$ is defined by $(L_x \varphi)(f) = \varphi(l_x f)$, $f \in L^\infty(G)$. Let LIM (or LIM(G)) be the set of left invariant means on $L^\infty(G)$, i.e., $\text{LIM} = \{\mu \in L^\infty(G)^*: \|\mu\| = 1, \mu \geq 0 \text{ and } L_x \mu = \mu \text{ for } x \in G\}$. G is said to be amenable if $\text{LIM} \neq \emptyset$. Every compact group is amenable since $\lambda \in \text{LIM}$.

Let $\mathfrak{D}$ be the maximal ideal space of $L^\infty(G)$. With the Gelfand topology, $\mathfrak{D}$ is a compact Hausdorff space. The Gelfand transform $\hat{\cdot}$ is an isometry of $L^\infty(G)$ onto $C(\mathfrak{D})$, the algebra of real-valued continuous functions on $\mathfrak{D}$ with supremum norm. Note that if $\theta \in \mathfrak{D}$ then $L_x \theta \in \mathfrak{D}$ and each $\mu \in \text{LIM}$ can be identified with a G -invariant probability measure $\hat{\mu}$ on $\mathfrak{D}$: $\hat{\mu}(\hat{f}) = \mu(f)$, $f \in L^\infty(G)$. For $\theta \in \mathfrak{D}$, let $\rho_\theta: L^\infty(G) \rightarrow l^\infty(G)$ be defined by

$$(\rho_\theta f)(x) = \theta(l_x f), \quad x \in G.$$

Here $l^\infty(G) = L^\infty(G_d)$ where G_d is the discrete version of G . Note that $\|\rho_\theta\| \leq 1$, $\rho_\theta \geq 0$ and $\rho_\theta(l_x f) = l_x(\rho_\theta f)$, for $x \in G$ and $f \in L^\infty(G)$.

The purpose of this paper is to prove the following.

THEOREM. *Let G be a σ -compact locally compact group. Then the following conditions are equivalent:*

- (1) G_d is amenable;
- (2) G is amenable and, for each $f \in L^\infty(G)$,

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$$\sup\{\mu(f): \mu \in \text{LIM}\} = \inf_A \text{ess sup}_x \left\{ \frac{1}{n} \sum_{i=1}^n f(a_i x) \right\},$$

where the infimum is taken over all finite subsets $A = \{a_1, \dots, a_n\}$ of G ;

(3) for each $\theta \in \mathfrak{D}$, the subspace $\{\rho_\theta f: f \in L^\infty(G)\}$ of $L^\infty(G)$ has a left invariant mean;

(4) for each permanently positive (PP) set $E \subset G$, $\|\chi_E - h\|_\infty \geq 1$ if $h \in H$ = the linear span of $\{l_x f - f: f \in L^\infty(G), x \in G\}$.

PP sets were first defined by Rudin [7]. Recall that a set $E \subset G$ is PP if it is Borel measurable and $\lambda(x_1 E \cap \dots \cap x_k E) > 0$ for any given $x_1, \dots, x_k \in G$.

Let

$$\text{TLIM} = \{ \mu \in L^\infty(G)^*: \|\mu\| = 1, \mu \geq 0$$

$$\text{and } \mu(\xi * f) = \mu(f) \text{ if } f \in L^\infty(G), \xi \in L^1(G), \|\xi\|_1 = 1, \xi \geq 0 \},$$

the set of topologically left invariant means on $L^\infty(G)$. It is known that $\text{TLIM} \subset \text{LIM}$ and if G is amenable then $\text{TLIM} \neq \emptyset$, cf. [2]. Granirer [1] and Rudin [7] proved, independently, that if G is nondiscrete and G_d is amenable then $\text{LIM} \supsetneq \text{TLIM}$. In particular, if G is an infinite compact group and G_d is amenable then $\text{LIM} \supsetneq \{\lambda\} = \text{TLIM}$. It is still unknown whether $\text{LIM} \supsetneq \{\lambda\}$ for every infinite compact group.

In [1], Granirer proved that (1) $\Rightarrow$ (2) and that if G is a nondiscrete group with (2) then $\text{LIM} \supsetneq \text{TLIM}$. He conjectured that (2) does not hold true in general if G_d is not amenable. Our (2) $\Rightarrow$ (1) confirms his conjecture. In [7], Rudin proved that (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) and that $\text{LIM} \supsetneq \text{TLIM}$ if G is nondiscrete and if either (3) or (4') holds where (4') is:

(4') If $h \in H$ then, for each open dense set $U \subset G$, $\|\chi_U - h\|_\infty \geq 1$.

He then suggested that (3) and (4') might hold true for every compact group. Our theorem shows that if G_d is not amenable then both (3) and (4) fail for G . Since every open dense set is PP, (4) $\Rightarrow$ (4'). We are unable to decide whether (4') $\Rightarrow$ (4) or whether (4') holds for every compact group.

In [7], Rudin raised the question whether functions of the form $\rho_\theta f$ are Borel measurable. Wells [8] employed a nice Baire categorical argument to show that for every infinite compact metric group G there exist $\theta \in \mathfrak{D}$ and $f \in L^\infty(G)$ such that $\rho_\theta f \notin L^\infty(G)$. Rosenblatt [5] elaborated Wells' argument to obtain a stronger result: If G is a nondiscrete metric group then for each $S \subset L^\infty(G)$, $\text{card } S \leq c$, there exists $\theta \in \mathfrak{D}$ such that $\rho_\theta(L^\infty(G)) \supset S$. As a consequence, he gave the following:

THEOREM A (ROSENBLATT). *Let G be a nondiscrete σ -compact locally compact group. Then G_d is amenable if and only if every nonempty closed invariant subset of $\mathfrak{D}$ admits a G -invariant probability measure.*

Our proof depends on this theorem of Rosenblatt.

PROOF OF THE THEOREM. Since (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are known, it remains to show that (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1).

(2) $\Rightarrow$ (4). Let E be a PP set in G . Then, for each finite set $A = \{a_1, \dots, a_n\} \subset G$,

$$\text{ess sup}_x \left\{ \frac{1}{n} \sum_{i=1}^n \chi_E(a_i x) \right\} = 1.$$

Therefore, by (2), there exists $\mu \in \text{LIM}$ such that $\mu(\chi_E) = 1$. (Recall that LIM is w^* -compact.) So, for each $h \in H$, $\mu(\chi_E - h) = 1$. Since $\|\mu\| = 1$, $\|\chi_E - h\|_\infty > 1$.

(3) $\Rightarrow$ (4). Let E be a PP set in G . As in [7], note that $I = \{f \in L^\infty(G) : \text{the zero set of } f \text{ is contained in } G \setminus E\}$ is a proper ideal. Therefore there exists $\theta \in \mathfrak{D}$ such that $\theta = 0$ on I . It implies that $\rho_\theta(\chi_E) \equiv 1$. By assumption, $\rho_\theta(L^\infty(G))$ has a left invariant mean ν . Let $h = \sum_{i=1}^n (l_{a_i} f_i - f_i) \in H$. Then

$$\rho_\theta(\chi_E - h) = 1 - \sum_{i=1}^n (l_{a_i}(\rho_\theta f_i) - (\rho_\theta f_i)).$$

So $\nu(\rho_\theta(\chi_E - h)) = 1$. Since $\|\rho_\theta(\chi_E - h)\|_\infty \leq \|\chi_E - h\|_\infty$ and $\|\nu\| = 1$, $\|\chi_E - h\|_\infty \geq 1$.

(4) $\Rightarrow$ (1). For each Borel set B in a locally compact group G there exists a closed-open set $\hat{B} \subset \mathfrak{D}$ such that $\hat{\chi}_B = \chi_{\hat{B}}$; in fact, $\hat{B} = \{\theta \in \mathfrak{D} : \theta(\chi_B) = 1\}$. Sets of the form $\hat{B}$ form a topological basis for $\mathfrak{D}$. It is easy to check that $(B_1 \cap B_2)^\wedge = \hat{B}_1 \cap \hat{B}_2$, $(B_1 \cup B_2)^\wedge = \hat{B}_1 \cup \hat{B}_2$ and $(xB_1)^\wedge = L_x(\hat{B}_1)$ if B_1 and B_2 are Borel sets in G and $x \in G$.

Now assume that (4) holds for G . If E is a PP set then by the Hahn-Banach theorem there exists $\mu \in \text{LIM}$ such that $\mu(\chi_E) = 1$ or, equivalently, $\hat{\mu}(\hat{E}) = 1$, i.e., $\text{supp } \hat{\mu} \subset \hat{E}$.

Let K be a nonempty closed G -invariant subset of $\mathfrak{D}$. Let

$$\mathfrak{E} = \{E : E \text{ a Borel subset of } G \text{ and } \hat{E} \supset K\}.$$

Then $\cap \{\hat{E} : E \in \mathfrak{E}\} = K$ and $\mathfrak{E}$ is a downward directed set. If $E \in \mathfrak{E}$ and $x_1, \dots, x_n \in G$ then

$$\begin{aligned} (x_1 E \cap \dots \cap x_n E)^\wedge &= L_{x_1} \hat{E} \cap \dots \cap L_{x_n} \hat{E} \\ &\supset L_{x_1} K \cap \dots \cap L_{x_n} K = K. \end{aligned}$$

So $\lambda(x_1 E \cap \dots \cap x_n E) > 0$ and hence E is a PP set. By the last paragraph, for each $E \in \mathfrak{E}$ we may choose $\mu_E \in \text{LIM}$ such that $\text{supp } \hat{\mu}_E \subset \hat{E}$. Let μ be a w^* -limit point of the net $\{\mu_E\}_{E \in \mathfrak{E}}$. Since LIM is w^* -compact, $\mu \in \text{LIM}$. If $\theta \in \mathfrak{D} \setminus K$, then there exists $E \in \mathfrak{E}$ such that $\theta \notin \hat{E}$. Therefore $\mu(\mathfrak{D} \setminus \hat{E}) = 0$ and $\mathfrak{D} \setminus \hat{E}$ is a neighborhood of θ . So $\theta \notin \text{supp } \mu$ and hence $\text{supp } \mu \subset K$. By Theorem A of Rosenblatt quoted above we conclude that G_d is amenable.

REMARKS. (1) By applying Rosenblatt's result it is possible to prove that (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (1) directly. Conceivably there is a direct proof of the fact that (4) $\Rightarrow$ (2) without applying his result.

(2) For a discrete group G , it is easy to see that a set $E \subset G$ is PP if and only if it is left thick in the sense of Mitchell [3]. Therefore, the following statement can be considered as a generalization of his Theorem 7: *Let G be a locally compact group, G_d amenable. Then a Borel set $E \subset G$ is PP if and only if there exists $\mu \in \text{LIM}$ such that $\mu(\chi_E) = 1$.*

(3) Assume that G is a σ -compact nondiscrete group. Then $P = \bigcap \{ \hat{U} : \hat{U} \text{ open dense in } G \}$ is a nonempty G -invariant closed subset of $\mathfrak{D}$. Therefore if G_d is amenable then there exists $\mu \in \text{LIM}$ such that $\hat{\mu}(P) = 1$. In particular, $\mu(\chi_U) = 1$ for each open dense set $U \subset G$. (Note that it is known that for each $\epsilon > 0$ there exists an open dense set U with $\lambda(U) < \epsilon$; see [1, Proposition 2].) If we assume further that G is compact then $U \cap E$ is PP if U is open dense and E is PP, cf. [7]. Let K be a minimal G -invariant subset of $\mathfrak{D}$, i.e., K is nonempty, closed and G -invariant and is minimal with respect to these properties. Then

$$(\bigcap \{ \hat{E} : E \text{ a Borel set, } \hat{E} \supset K \}) \cap (\bigcap \{ \hat{U} : U \text{ open dense in } G \})$$

is nonempty, i.e., $K \cap P \neq \emptyset$. By minimality of K , $K \subset P$. (This fact is also contained in the proof of Proposition 4.8 in [6].) Note that Rosenblatt [5] has shown that $\mathfrak{D}$ has at least 2^c minimal G -invariant sets.

(4) For an infinite compact group G , the condition that G_d is amenable is not a necessary condition for $\text{LIM} \supsetneq \{ \lambda \}$. First let us give the following.

LEMMA. *Let G be a compact group and N a closed normal subgroup of G . Then there is a continuous affine embedding of $\text{LIM}(G/N)$ into $\text{LIM}(G)$.*

PROOF. If $f \in L^\infty(G)$ let $\tilde{f} \in L^\infty(G/N)$ be defined by

$$\tilde{f}(\dot{x}) = \int_N f(xt) d\lambda_N(t), \quad x \in G.$$

Here $\dot{x}$ is the coset xN and λ_N is the normalized Haar measure of N . It is easy to check that $\tilde{f}$ is indeed a well-defined function in $L^\infty(G/N)$ and that $\|\tilde{f}\|_\infty \leq \|f\|_\infty$.

If $\nu \in \text{LIM}(G/N)$, define $\tilde{\nu} \in L^\infty(G)^*$ by $\tilde{\nu}(f) = \nu(\tilde{f})$, $f \in L^\infty(G)$. It is easily checked that $\|\tilde{\nu}\| = 1$, $\tilde{\nu} \geq 0$ and, if $x \in G$ and $f \in L^\infty(G)$,

$$\tilde{\nu}(l_x f) = \nu((l_x f)^\sim) = \nu(l_x(\tilde{f})) = \nu(\tilde{f}) = \tilde{\nu}(f).$$

Therefore, $\tilde{\nu} \in \text{LIM}(G)$. Since $f \rightarrow \tilde{f}$ sends $L^\infty(G)$ onto $L^\infty(G/N)$, the mapping $\nu \rightarrow \tilde{\nu}$ is one-to-one.

By the above lemma, we know that $\text{LIM}(G) \supsetneq \{ \lambda \}$ if $G = \text{SU}(2) \times \mathbb{T}$ (direct product). The following example is perhaps more interesting.

EXAMPLE. A theorem of Iwasawa states that a free group is residually a finite p -group (for all primes p) (cf. [4, p. 117]). Therefore, if F is the free group on two generators then there exists a sequence of normal subgroups $\{F_n\}$ of F such that $F \supsetneq F_1 \supsetneq F_2 \supsetneq \dots$, $\bigcap_{n=1}^\infty F_n = \{1\}$, and for each n , $F/F_n = G_n$ is a finite p -group. Then F can be considered as a subgroup of $G = \prod_{n=1}^\infty G_n$ (complete product). Hence, with product topology G is compact

and G_d is nonamenable. We claim that $\text{LIM}(G) \supsetneq \{\lambda\}$. Indeed for each n , choose a (proper) maximal subgroup N_n of G_n . Then $[G_n : N_n] = p$ and N_n is normal in G . Let $N = \prod_{n=1}^{\infty} N_n$. Then N is a compact normal subgroup of G and $G/N \cong \prod_{n=1}^{\infty} (G_n/N_n)$ is an infinite abelian compact group. By the above lemma, $\text{LIM}(G) \supsetneq \{\lambda\}$.

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