INTERPOLATION IN H^p -SPACES

KNUT ØYMA

ABSTRACT. We prove that when we solve the classical interpolation problem in H^p , we can also interpolate the first few derivatives at each point. We also study interpolating functions of minimal norm.

Let $D = \{z \mid |z| < 1\}$. For $0 < p < \infty$ let H^p be the set of functions analytic in D such that

$$\lim_{r \rightarrow 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

and let H^∞ be the set of all bounded analytic functions in D . For $1 \leq p < \infty$, H^p is a Banach space, and, for $0 < p < 1$, H^p is an F -space. For properties of the H^p -spaces, see [3].

A sequence $\{z_n\}$ in D is called uniformly separated if

$$\inf_n \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| = \delta > 0.$$

Uniformly separated sequences play a dominant role in interpolation theory.

Let T_p be the operator from H^p into the space of complex sequences defined by $T_p(f) = \{f(z_n)(1 - |z_n|^2)^{1/p}\}$. In their paper [8] Shapiro and Shields proved that, for $1 \leq p \leq \infty$, $T_p(H^p) = l^p$ if and only if $\{z_n\}$ is uniformly separated. For $p = \infty$ this had already been proved by Carleson [2]. Shortly after Shapiro and Shields published their result, Kabaila proved the theorem for $0 < p < 1$ [7].

Let N be a positive integer. Define

$$T_{p,N}(f) = \left(\left\{ f(z_n)(1 - |z_n|^2)^{1/p} \right\}, \left\{ f'(z_n)(1 - |z_n|^2)^{1/p+1} \right\}, \dots, \right. \\ \left. \left\{ f^{(N-1)}(z_n)(1 - |z_n|^2)^{1/p+N-1} \right\}_{n=1}^\infty \right).$$

We know that f' grows faster than f by a factor $(1 - |z|)^{-1}$. For precise interpretations of this see Chapter 5 of [3]. Therefore the following result is in a way natural.

THEOREM 1. *If $\{z_n\}$ is uniformly separated and $0 < p \leq \infty$, $T_{p,N}(H^p) = (l^p)^N$.*

We first prove the theorem for $p = \infty$. Note that the Cauchy formula gives the estimate

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$$|f^{(k)}(z)| \leq k! \|f\| (1 - |z|)^{-k}.$$

We are going to prove that if the numbers $w_{n,k}$ satisfies $|w_{n,k}| \leq C(1 - |z_n|)^{-k}$, there is an f in H^∞ such that $f^{(k)}(z_n) = w_{n,k}$ for all n and $k = 0, 1, \dots, N - 1$. The proof uses induction on N . For $N = 1$ this is the Carleson theorem. Assume we have found $h \in H^\infty$ such that $h^{(k)}(z_n) = w_{n,k}$ for all n and $k < K$. Let

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \cdot \frac{z_n - z}{1 - \bar{z}_n z}.$$

We have to prove that there is a function $g \in H^\infty$ such that $(gB^K)^{(K)}(z_n) = w_{n,K} - h^{(K)}(z_n)$. Then $h + B^K g$ solves our problem. Let

$$\delta_n = \prod_{k \neq n} \frac{|z_k|}{z_k} \cdot \frac{z_k - z_n}{1 - \bar{z}_k z_n}.$$

Calculation shows that

$$(B^K g)^{(K)}(z_n) = K! g(z_n) \left(\delta_n \frac{|z_n|}{z_n} \cdot \frac{1}{1 - |z_n|^2} \right)^K.$$

The theorem is therefore proved for $p = \infty$ if we can find $g \in H^\infty$ satisfying

$$g(z_n) = \frac{1}{K!} \left(\delta_n \frac{|z_n|}{z_n} \cdot \frac{1}{1 - |z_n|^2} \right)^{-K} (w_{n,K} - h^{(K)}(z_n)).$$

These numbers are bounded by the Cauchy estimate and the hypothesis; hence Carleson's theorem proves that $(l^\infty)^N \subseteq T_{\infty,N}(H^\infty)$. The opposite inclusion follows from Cauchy's estimate. We remark that the norm of f depends only on C , N and δ .

Our next goal is to prove:

LEMMA 1. *If $1 \leq p \leq 2$, $f \in H^p$ and if $\{z_n\}$ is uniformly separated, then $\{f^{(K)}(z_n)(1 - |z_n|^2)^{1/p+K}\} \in l^p$ for $K = 0, 1, \dots$ and*

$$\left\| \{f^{(K)}(z_n)(1 - |z_n|^2)^{1/p+K}\} \right\|_p \leq A_{p,K} \|f\|_p.$$

The existence of $A_{p,K}$ follows from the closed graph theorem when the first part is proved. Let

$$\begin{aligned} {}_k B(z) &= \prod_{n \neq k} \frac{|z_n|}{z_n} \cdot \frac{z_n - z}{1 - \bar{z}_n z}, & B_N(z) &= \prod_{n=1}^N \frac{|z_n|}{z_n} \cdot \frac{z_n - z}{1 - \bar{z}_n z}, \\ {}_k B_N(z) &= \prod_{n \neq k}^N \frac{|z_n|}{z_n} \cdot \frac{z_n - z}{1 - \bar{z}_n z} & (k \leq N) \end{aligned}$$

and let ${}_N \delta_k = {}_k B_N(z_k)$. The proof needs:

LEMMA 2. *If $\{z_k\}$ is uniformly separated, then*

$$|(1/{}_k B)^{(n)}(z_k)| \leq C_n (1 - |z_k|)^{-n}.$$

The estimate remains true if ${}_k B$ is replaced by $({}_k B)^m$ or $({}_k B_N)^m$ if C_n is replaced by $C_{n,m}$.

The proof of Lemma 2 uses induction on n . The lemma is true for $n = 0$ because $\{z_k\}$ is uniformly separated. The induction step is carried out by differentiating the identity $1 = (1/{}_k B) \cdot {}_k B$. The details are straightforward and are omitted.

We now prove Lemma 1 for $p = 1$. For $K = 0$ the lemma follows from the Shapiro-Shields theorem. Assume it has been proved for $K - 1$. Let $w_{n,k}$ satisfy

$$|w_{n,k}| < C(1 - |z_k|)^{-k} \quad (*)$$

and let $f \in H^\infty$ satisfy $f^{(k)}(z_n) = w_{n,k}$ for all n and $k = 0, \dots, K$. The norm of f depends only on K , C and δ . Let γ_n be a small circle surrounding z_n . By the duality relation [3, p. 130] we have:

$$\begin{aligned} \|f\| &\geq \inf_{h \in H^\infty} \|f + B_N^{K+1} \cdot h\| = \sup_{g \in \text{ball } H^1} \left| \frac{1}{2\pi} \int_T \frac{f(z)}{B_N^{K+1}} g(z) dz \right| \\ &= \sup_{g \in \text{ball } H^1} \left| \sum_{k=1}^N \frac{1}{2\pi} \int_{\gamma_k} \frac{f(z)g(z)(1 - \bar{z}_k z)^{K+1}}{({}_k B_N(z))^{K+1}(z - z_k)^{K+1}(-1)^{K+1}} \left(\frac{z_k}{|z_k|}\right)^{K+1} dz \right| \\ &= \sup_{g \in \text{ball } H^1} \left| \frac{i}{K!(-1)^{K+1}} \sum_{k=1}^N \left(\frac{f(z)g(z)(1 - \bar{z}_k z)^{K+1}}{{}_k B_N^{K+1}(z)} \left(\frac{z_k}{|z_k|}\right)^{K+1} \right)^{(K)} (z_k) \right|. \end{aligned}$$

We have

$$(f_1 \cdot f_2 \cdot f_3 \cdot f_4)^{(K)} = \sum_{\substack{r+s+t+u=K \\ r,s,t,u \geq 0}} a_{r,s,t,u} f_1^{(r)} f_2^{(s)} f_3^{(t)} f_4^{(u)}$$

where the $a_{r,s,t,u}$ are positive numbers. Therefore our supremum equals

$$\begin{aligned} &\sup_{g \in \text{ball } H^1} \frac{1}{K!} \left| \sum_{\substack{r+s+t+u=K \\ r,s,t,u \geq 0}} \sum_{k=1}^N a_{r,s,t,u} \left(\frac{1}{{}_k B_N^{K+1}}\right)^{(r)} (z_k) g^{(s)}(z_k) \right. \\ &\quad \cdot f^{(t)}(z_k) ([1 - \bar{z}_k z]^{K+1})^{(u)}(z_k) \left. \right| \\ &= \frac{1}{K!} \sup_{g \in \text{ball } H^1} \left| \sum_{k=1}^N a_{0,K,0,0} \left(\frac{1}{N\delta_k}\right)^{K+1} g^{(K)}(z_k) w_{k,0} (1 - |z_k|^2)^{K+1} \right. \\ &\quad + \sum_{\substack{r+s+t+u=K \\ s < K}} \sum_{k=1}^N a_{r,s,t,u} \left(\frac{1}{{}_k B_N^{K+1}}\right)^{(r)} \\ &\quad \cdot (z_k) g^{(s)}(z_k) w_{k,t} [1 - \bar{z}_k z]^{(K+1)(u)}(z_k) \left. \right|. \end{aligned}$$

By Lemma 2 the term of the last sum is dominated by

$$\begin{aligned} a_{r,s,t,u} C_{r,K+1} (1 - |z_k|)^{-r} g^{(s)}(z_k) C (1 - |z_k|)^{-t} C' (1 - |z_k|)^{K+1-u} \\ = c'' g^{(s)} (1 - |z_k|)^{s+1}. \end{aligned}$$

Hence by the induction hypothesis, the last sum is bounded independent of N .

Taking the supremum over all $w_{n,k}$ satisfying (*) and letting $N \rightarrow \infty$, Lemma 1 is proved for $p = 1$.

To prove the lemma for $p = 2$ we need the Hilbert spaces $A^{2,\alpha}$. If f is analytic in D , we define its $A^{2,\alpha}$ norm by

$$\|f\|_{2,\alpha}^2 = \frac{1}{\pi} \int_D |f(z)|^2 (1 - |z|^2)^\alpha dA$$

where dA is the ordinary area measure. Using Parseval's formula we can prove

$$f \in A^{2,\alpha} \Leftrightarrow \sum |a_n|^2 (n+1)^{-1-\alpha} < \infty. \quad (**)$$

This is easy when α is a natural number, and that is enough for us, but the formula holds for every α such that $0 \leq \alpha < \infty$ (see [6]). For $f \in A^{2,\alpha}$ we have

$$\sum_{n=1}^{\infty} (1 - |z_n|^2)^{2+\alpha} |f(z_n)|^2 \leq K_\alpha \|f\|_{2,\alpha}^2 \quad (***)$$

when $\{z_n\}$ is uniformly separated. This is proved as follows: Let

$$D_n = \left\{ z: |z_n - z| < \frac{1}{8} \delta (1 - |z_n|^2) \right\}.$$

Since $|(z_n - z_m)/(1 - \bar{z}_n z_m)| \geq \delta$, we easily prove that the discs D_n are disjoint. By the subharmonicity of $|f(z)|^2$ we obtain by integrating in polar coordinates

$$|f(z_n)|^2 (1 - |z_n|^2)^2 \leq \frac{64\pi}{\delta^2} \int_{D_n} |f(z)|^2 dA.$$

Hence by geometric considerations

$$\begin{aligned} |f(z_n)|^2 (1 - |z_n|^2)^{2+\alpha} &\leq \frac{64\pi}{\delta^2} \int_{D_n} |f(z)|^2 (1 - |z_n|^2)^\alpha dA \\ &\leq 2^\alpha \frac{64\pi}{\delta^2} \int_{D_n} |f(z)|^2 (1 - |z|^2)^\alpha dA. \end{aligned}$$

(***) is now obtained by summing.

Using Parseval's formula again, we see that $f \in H^2 \Rightarrow f^{(K)} \in A^{2,2K-1}$. Then (***) gives

$$\sum_{n=1}^{\infty} (1 - |z_n|^2)^{2K+1} |f^{(K)}(z_n)|^2 < \infty.$$

This proves the lemma for $p = 2$. It is now easy to prove Lemma 1 for $1 < p < 2$.

Let μ be the measure which assigns to the nonnegative integer n the mass $(1 - |z_n|)^2$. Let

$$T(f) = \left\{ f^{(K)}(z_n)(1 - |z_n|^2)^K \right\}_{n=1}^{\infty}.$$

We have proved that T maps H^1 into $L^1(d\mu)$ and H^2 into $L^2(d\mu)$. By Theorem 3.9 of [10], T is also of type (p, p) , $1 < p < 2$, for all polynomials. They are dense in H^p , hence one easily proves that T maps H^p into $L^p(d\mu)$. This proves Lemma 1. Lemma 1 says that $T_{p,N}(H^p) \subseteq (l^p)^N$. To complete the proof of Theorem 1 for $1 < p < 2$ copy the proof of the same theorem for $p = \infty$. The details are again omitted.

We now prove the theorem for $2 < p < \infty$. We first prove that Lemma 1 also holds for these values of p . This is done by a duality argument and induction on K . For $K = 0$ the result is well known. Assume the lemma holds for $K - 1$. Let $f \in H^p$, $\|f\|_p = 1$, $g \in H^q$, $\|g\|_q = 1$ where $1/p + 1/q = 1$. Then $fg \in H^1$. By the Hölder inequality and Lemma 1 for $p = 1$ we have

$$\begin{aligned} 1 &\geq \|fg\|_1 \geq (A_{1,K})^{-1} \left| \sum_{n=1}^{\infty} (fg)^{(K)}(z_n)(1 - |z_n|^2)^{K+1} \right| \\ &= (A_{1,K})^{-1} \left| \sum_{n=1}^{\infty} \sum_{k=0}^K \binom{K}{k} f^{(k)}(z_n) g^{(K-k)}(z_n) (1 - |z_n|^2)^{K+1} \right| \\ &= (A_{1,K})^{-1} \left| \sum_{n=1}^{\infty} \sum_{k=0}^K \binom{K}{k} f^{(k)}(z_n) (1 - |z_n|^2)^{1/p+k} \right. \\ &\quad \left. \cdot g^{(K-k)}(z_n) (1 - |z_n|^2)^{1/q+(K-k)} \right| \\ &= (A_{1,K})^{-1} \left| \sum_{n=1}^{\infty} \left(f^{(K)}(z_n) (1 - |z_n|^2)^{1/p+K} g(z_n) (1 - |z_n|^2)^{1/q} \right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{K-1} \binom{K}{k} f^{(k)}(z_n) (1 - |z_n|^2)^{1/p+k} \right. \right. \\ &\quad \left. \left. \cdot g^{(K-k)}(z_n) (1 - |z_n|^2)^{1/q+(K-k)} \right) \right|. \end{aligned}$$

By Lemma 1, the induction hypothesis and the Hölder inequality the last sum is bounded independent of f and g when their norms do not exceed 1. Hence the sum

$$\left| \sum_{n=1}^{\infty} f^{(K)}(z_n) (1 - |z_n|^2)^{1/p+K} g(z_n) (1 - |z_n|^2)^{1/q} \right| \leq D \|g\|_q.$$

The Hahn-Banach theorem and the Shapiro-Shield's interpolation theorem now prove Lemma 1 for $2 < p < \infty$. The proof of Theorem 1 for $2 < p < \infty$ is completed just as in the case $1 < p < 2$. It remains to prove Theorem 1 for $0 < p < 1$. For that we proceed along a different line. We call $f \in H^p$ good of order K if

$$\{f^{(k)}(z_n)(1 - |z_n|^2)^{1/p+k}\} \in l^p \quad \text{for } k = 0, \dots, K.$$

Let $K > N$. We first prove that $(l^p)^N \subseteq T_{p,N}\{f \in H^p | f \text{ is good of order } K\}$. Let $N = 1$, and let $\{w_n\} \in l^p$ be arbitrary. Let $b_m(z) \in H^\infty$ satisfy $b_m(z_m) = 1$, $b_m(z_n) = 0$ for $m \neq n$. $b^{(k)}(z_n) = 0$ for all n and $k = 1, \dots, N$. $\|b_m\| < C$, where C is independent of m . Let

$$f(z) = \sum_{n=1}^{\infty} (1 - |z_n|^2)^{1/p} w_n b_n(z) (1 - \bar{z}_n z)^{-2/p}.$$

Exactly as in [3] we prove that $f \in H^p$ and $T_{p,1}(f) = \{w_n\}$. The induction step is carried out in the same way as Theorem 1 for $p = \infty$. This proves that $(l^p)^N \subseteq T_{p,N}(H^p)$. The opposite inclusion, that is, every $f \in H^p$ is good of order N , is proved by induction. For $N = 1$ this is Kabaila's theorem.

Assume every $f \in H^p$ is good of order $K - 1$. Choose $g \in H^p$ good of order K such that $g^{(k)}(z_n) = f^{(k)}(z_n)$ for all n and $k = 0, \dots, K - 1$. This is possible since f is good of order $K - 1$. Let $h = f - g$. It is enough to prove that

$$\{h^{(K)}(z_n)(1 - |z_n|^2)^{1/p+K}\} \in l^p.$$

h has a zero of order at least K at each z_n . Hence $h = B^K \cdot G$. But

$$\begin{aligned} (B^K \cdot G)^{(K)}(z_n)(1 - |z_n|^2)^{1/p+K} \\ = K! G(z_n) \left(\delta_n \frac{|z_n|}{z_n} \cdot \frac{1}{1 - |z_n|^2} \right)^K (1 - |z_n|^2)^{1/p+K} \\ = K! \left(\delta_n \frac{|z_n|}{z_n} \right)^K G(z_n) (1 - |z_n|^2)^{1/p}. \end{aligned}$$

This sequence lies in l^p since $G \in H^p$. This completes the proof of Theorem 1.

Let $x \in (l^p)^N$ and let $f \in H^p$ satisfy $x = T_{p,N}(f)$. By a normal family argument we may assume that $\|f\|_p$ is minimal. Such an f is called extremal. In [1] Akutowitch and Carleson studied extremal functions for various Hilbert spaces of analytic functions. A relevant result also appears in [9] and [11].

By the duality relation we have, for $1 \leq p \leq \infty$,

$$\|f\|_p = \inf_{g \in H^p} \|f + B^N g\|_p = \sup_{\substack{g \in H^q \\ \|g\|_q = 1}} \left| \frac{1}{2\pi} \int_T \frac{fg}{B^N} dz \right|.$$

Our results will follow from this relation. Let E be the (open) set of the unit circle where $\{z_n\}$ does not cluster. We are going to prove:

THEOREM 2. *If $1 \leq p < \infty$, an extremal function is unique. If $p = 1, 2$, the extremal function continues analytically across $\bar{E}$. If $1 < p < 2$ or $2 < p < \infty$, the extremal function has analytic continuation across E with possible exception of a point set without cluster points in E .*

PROOF. The uniqueness follows from Theorem 8.1 in [3]. Let us first consider the case $p = 1$. By the same theorem

$$\|f\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{B^N(e^{i\theta})} g(e^{i\theta}) e^{i\theta} d\theta$$

for some $g \in H^\infty$ of norm 1. Hence g is an inner function and $f(e^{i\theta}) \cdot g(e^{i\theta}) e^{i\theta} / B^N(e^{i\theta}) \geq 0$ a.e. Every point of E has a neighborhood where $f(z)g(z)z / B^N(z)$ has a harmonic majorant. This shows that $f(e^{i\theta}) \cdot g(e^{i\theta}) e^{i\theta} / B^N(e^{i\theta}) \geq 0$ everywhere on E , hence $(f(z)g(z) / B^N(z)) \cdot z$ has analytic continuation across E . This is also true for $f(z)g(z)$, hence also for $f(z)$ since $g(z)$ is inner (see [5, pp. 68–70]). For $1 < p < \infty$ we have by the same method:

$$\|f\|_p = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})g(e^{i\theta})}{B^N(e^{i\theta})} e^{i\theta} d\theta \quad (****)$$

for some $g \in H^q$ of norm 1.

The same argument proves that $f(z)g(z)$ has analytic continuation across E . Let $f = I_f F$ and $g = I_g G$ be the factorizations of f and g in inner and outer functions. The inner part of fg is $I_f I_g$, and this function has analytic continuation across E . Hence the same is true for I_f . Equality in (****) shows that $|g(e^{i\theta})| = |f(e^{i\theta})|^{p/q}$ a.e. on T . The outer part of fg has also analytic continuation across E . This function equals

$$\begin{aligned} F(z)G(z) &= F(z) \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |g(e^{i\theta})| d\theta \right\} \\ &= F(z) \cdot F^{p/q}(z) = F^p(z). \end{aligned}$$

Hence $F(z)$ has analytic continuation across the part of E where $F^p(z) \neq 0$. This completes the proof of Theorem 2 except for the case $p = 2$. Since $(f(z)g(z) \cdot z) / B^N(z) \geq 0$ on E , a zero of $FG = F^2$ has to be of even order. Taking the square root we prove the result for $p = 2$.

REMARK. For $N = 1$ the result for $p = 2$ is in [1]. We could have used the same method for arbitrary N . Our result for $N = 1 = p$ is announced there.

The case $p = \infty$ is more complicated. In this case an extremal function need not be unique (see [11]) and even if it is unique, f may have the unit circle as its natural boundary (see [9]). However, we intend to prove

THEOREM 3. *If $w_{n,k}$ satisfies $w_{n,k} = o(1 - |z_n|)^{-k}$, there is a unique $f \in H^\infty$ of minimal norm such that $f^{(k)}(z_n) = w_{n,k}$ for all n and $k = 0, \dots, N$. f is a complex constant times an inner function with analytic continuation across E . If in addition $z_n \rightarrow 1$ nontangentially, f is a complex constant times a Blaschke product.*

The proof of the first part of this theorem follows the same lines as the proof of Theorem 2 of [11] and is therefore omitted. To prove the second part

of Theorem 3 we observe that $f(z) = \lambda B^*(z) \exp\{(z+1)/(z-1)\gamma\}$ where $\gamma \geq 0$. We want to prove that $\gamma = 0$. If $\gamma > 0$, $f^{(k)}(z) = O(1-z)^m$ for every m and k if z lies in a Stolz angle. Hence we may assume that $w_{n,k} = o(1-z_n)^m$ for every m .

Let Ω be the maximal star shaped subset w.r.t. 0 of the set $\{z: |B(z)| < \delta^{-1}, |z| < 2\}$. We recall that $\delta = \inf_n |B_n(z_n)|$. $\{z_n\}$ is an interpolating sequence for $H^\infty(\Omega)$ (See Lemma 1 of [4]). We want to prove that there is a function $h(z)$ such that $h^{(k)}(z_n) = w_{n,k}$ such that $h(z) = (z-1)^2 g(z)$ where $g(z) \in H^\infty(\Omega)$. The proof of Theorem 1 for $p = \infty$ shows that if $\alpha_{n,k}$ is $O(1-|z_n|)^{-k}$, there is an interpolating function in $H^\infty(\Omega)$. Our $g(z)$ must satisfy

$$((z-1)^2 g(z))^{(k)}(z_n) = w_{n,k}.$$

From this we get the equations $g^{(k)}(z_n) = \alpha_{n,k}$. Since $w_{n,k} = O(1-z_n)^m$ for all m , the same is true for $\alpha_{n,k}$, hence $g(z)$ exists. The next step of the proof consists of proving that $h(z)/B^N(z)|_T \in C^1(T)$. This is done exactly in the same way as the proof for $N = 1$ in [11] and is therefore omitted. The last part of the proof is a direct copy for the proof of Theorem 3 in [11] and is also omitted.

In this paper we have interpolated the first few derivatives at each point. Here few means uniformly few. If we want to interpolate the first $\beta(n)$ derivatives at z_n and $\sup \beta(n) = \infty$, the problem is, of course, more complicated. A partial result for $p = \infty$ appears in [12].

REFERENCES

1. E. J. Akutowicz and L. Carleson, *The analytic continuation of interpolatory functions*, J. Analyse Math. 7 (1959/60), 223–247.
2. L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. 80 (1958), 921–930.
3. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
4. E. A. Heard and J. H. Wells, *An interpolation problem for subalgebras of H^∞* , Pacific J. Math. 28 (1969), 543–553.
5. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
6. C. Horowitz, *Zeros of functions in the Bergman spaces*, Ph.D. thesis, Univ. of Michigan, 1974.
7. V. Kabaila, *Interpolation sequences for H_p classes in the case $p < 1$* , Litovsk. Mat. Sb. 3 (1963), no. 1, 141–147. (Russian)
8. H. S. Shapiro and A. L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. 83 (1961), 513–532.
9. A. Stray and K. O. Øyma, *On interpolating functions with minimal norm*, Proc. Amer. Math. Soc. 68 (1978), 75–78.
10. A. Zygmund, *Trigonometric series*. vol. II, Cambridge Univ. Press, Cambridge, 1968.
11. K. O. Øyma, *Extremal interpolatory functions in H^∞* , Proc. Amer. Math. Soc. 64 (1977), 272–276.
12. ———, *Interpolation theorems for H^∞* , Preprint series for Agder Distrikthøgskole, 1975.

DEPARTMENT OF MATHEMATICS, AGDER DISTRIKTHØGSKOLE, KRISTIAN SAND, NORWAY