ROBERTSON'S CONJECTURE ON THE COEFFICIENTS OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. We use an inequality due to Lebedev and Milin to prove a conjecture made by M.S. Robertson on the coefficients of close-to-convex functions.

Let S denote the class of functions

$$f(z) = z + a_2 z^2 + \dots$$

that are regular and univalent in the unit disk $|z| < 1$. Also, let S^* denote the subclass of S consisting of functions which map the unit disk onto domains which are starlike with respect to the origin. A function $f(z)$ in S is close-to-convex if there exists a function $g(z)$ in S^* such that

$$\operatorname{Re} e^{i\alpha} \frac{zf'(z)}{g(z)} > 0,$$

for $|z| < 1$, with α real.

In this note, we prove a conjecture raised by M. S. Robertson in [5], namely,

$$|n|a_n| - m|a_m|| \leq |n^2 - m^2|$$

for all $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are close-to-convex.

THEOREM. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is close-to-convex, then*

$$|n|a_n| - m|a_m|| \leq |n^2 - m^2|$$

for all positive n and m . Strict inequality holds for all n and m , with $n \neq m$, unless $f(z)$ is a rotation of the Koebe function $z/(1-z)^2$.

COROLLARY 1. *For every close-to-convex function,*

$$|a_n| \leq n, \quad n = 2, 3, \dots, \tag{1}$$

$$-3 + 2/n \leq |a_n| - |a_{n-1}| \leq 2 - 1/n. \tag{2}$$

The first part of the corollary is due to M. Reade [4].

The proof of the theorem depends on a lemma due to T. H. MacGregor [2], concerning the class P of functions

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$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

analytic and of positive real part in $|z| < 1$.

LEMMA 1. Let $p \in P$, and let $\lambda_n \geq 0$. If $q(z) = \sum_{n=1}^{\infty} \lambda_n p_n z^n$ is analytic in $|z| < 1$ and $\operatorname{Re} q(z) \leq M$ for some positive M , then $\sum_{n=1}^{\infty} \lambda_n |P_n|^2 \leq 2M$.

COROLLARY TO THE LEMMA. For each $p \in P$ and for each finite sequence $\{\lambda_k\}_{k=1}^n$ with $\lambda_k \geq 0$, there exists a ζ with $|\zeta| = 1$ such that

$$\sum_{k=1}^n \lambda_k |p_k - \zeta^k|^2 \leq \sum_{k=1}^n \lambda_k.$$

PROOF OF THE COROLLARY. Apply the lemma to $q(z) = \sum_{k=1}^n \lambda_k p_k z^k$. This gives

$$\begin{aligned} \sum_{k=1}^n \lambda_k |p_k - \zeta^k|^2 &= \sum_{k=1}^n \lambda_k |p_k|^2 - 2 \operatorname{Re} q(\bar{\zeta}) + \sum_{k=1}^n \lambda_k \\ &\leq 2M - 2 \operatorname{Re} q(\bar{\zeta}) + \sum_{k=1}^n \lambda_k. \end{aligned}$$

Choosing ζ with $|\zeta| = 1$ so that $\operatorname{Re} q(\bar{\zeta}) = M = \max_{|z|=1} \operatorname{Re}\{q(z)\}$, we obtain the result.

LEMMA 2. If $\sum_{k=0}^{\infty} D_k z^k = \exp(\sum_{k=1}^{\infty} A_k z^k)$, and both functions are analytic in $|z| < 1$, then

$$\sum_{k=0}^{n-1} |D_k|^2 \leq n \exp \left\{ \sum_{k=1}^n \left(k |A_k|^2 - \frac{k^2 |A_k|^2}{n} - \frac{1}{k} \right) + 1 \right\},$$

with equality if and only if $A_k = c^k/k$, $|c| = 1$, $k = 1, 2, \dots, n-1$.

A proof of this so-called Lebedev-Milin inequality can be found in D. Aharonov's lecture notes [1, Theorem 1.2]. A similar proof can be found in Pommerenke's book [3, Lemma 3.3].

LEMMA 3. With the notations above, we have

$$\begin{aligned} \sum_{k=0}^{n-1} |D_k|^2 + \frac{|D_n|^2}{2} &\leq \frac{2n+1}{2} \exp \left\{ \sum_{k=1}^n \left(\frac{k^2}{n(2n+1)} + k - \frac{k^2}{n} \right) |A_k|^2 \right. \\ &\quad \left. - \sum_{k=1}^n \frac{1}{k} - \frac{1}{2n+1} + 1 \right\}, \end{aligned}$$

with equality if and only if $A_k = c^k/k$, $|c| = 1$, $k = 1, 2, \dots, n$.

PROOF. We imitate the proof as given in [1, Theorem 1.2]. Differentiating both sides of

$$\sum_{k=0}^{\infty} D_k z^k = \exp\left(\sum_{k=1}^{\infty} A_k z^k\right),$$

and comparing the coefficients of z^{n-1} , we get

$$nD_n = \sum_{k=1}^n kA_k D_{n-k}.$$

By the Cauchy-Schwarz inequality,

$$|D_n|^2 \leq \sum_{k=1}^n \frac{k^2 |A_k|^2}{n^2} \sum_{k=0}^{n-1} |D_k|^2.$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{n-1} |D_k|^2 + \frac{|D_n|^2}{2} &\leq \sum_{k=0}^{n-1} |D_k|^2 + \frac{1}{2} \sum_{k=1}^n \frac{k^2 |A_k|^2}{n^2} \sum_{k=0}^{n-1} |D_k|^2 \\ &= \left(1 + \frac{1}{2} \sum_{k=1}^n \frac{k^2 |A_k|^2}{n^2}\right) \sum_{k=0}^{n-1} |D_k|^2 \\ &= \frac{2n+1}{2n} \left(1 + \sum_{k=1}^n \frac{k^2 |A_k|^2}{n(2n+1)} - \frac{1}{2n+1}\right) \sum_{k=0}^{n-1} |D_k|^2 \\ &\leq \frac{2n+1}{2n} \exp\left\{\sum_{k=1}^n \frac{k^2 |A_k|^2}{n(2n+1)} - \frac{1}{2n+1}\right\} \sum_{k=0}^{n-1} |D_k|^2, \end{aligned}$$

where we applied the inequality $1 + x \leq e^x$ with

$$x = \sum_{k=1}^n \frac{k^2 |A_k|^2}{n(2n+1)} - \frac{1}{2n+1}.$$

Finally, applying Lemma 2 to estimate the last term of the above inequality, we get

$$\begin{aligned} \sum_{k=0}^{n-1} |D_k|^2 + \frac{|D_n|^2}{2} &\leq \frac{2n+1}{2} \exp\left\{\sum_{k=1}^n \left(\frac{k^2}{n(2n+1)} + k - \frac{k^2}{n}\right) |A_k|^2 \right. \\ &\quad \left. - \sum_{k=1}^n \frac{1}{k} - \frac{1}{2n+1} + 1\right\}. \end{aligned}$$

This completes the proof of the first part of the lemma.

The statement about the case of equality follows in the same way as in [1, Theorem 1.2], and we skip the proof.

LEMMA 4. If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is in S^* , for each positive integer n , there exists a ζ with $|\zeta| = 1$ such that

$$1 + \sum_{k=1}^{n-1} |b_{k+1} - \zeta b_k|^2 + \frac{|b_{n+1} - \zeta b_n|^2}{2} < n + \frac{1}{2}.$$

PROOF. If $g(z) \in S^*$, then it is well known that

$$\frac{zg'(z)}{g(z)} = p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

with $p(z)$ in P . Hence

$$\log \frac{g(z)}{z} = \sum_{n=1}^{\infty} \frac{p_n}{n} z^n.$$

Take a ζ on the unit circle, then

$$\begin{aligned} (1 - \zeta z) \frac{g(z)}{z} &= 1 + \sum_{k=1}^{\infty} (b_{k+1} - \zeta b_k) z^k \\ &= \sum_{k=0}^{\infty} D_k z^k, \text{ say.} \end{aligned}$$

On the other hand,

$$\begin{aligned} \log(1 - \zeta z) \frac{g(z)}{z} &= \sum_{k=1}^{\infty} \frac{(p_k - \zeta^k)}{k} z^k \\ &= \sum_{k=1}^{\infty} A_k z^k, \text{ say.} \end{aligned}$$

Then,

$$\sum_{k=0}^{\infty} D_k z^k = \exp \left(\sum_{k=0}^{\infty} A_k z^k \right).$$

By Lemma 3,

$$\begin{aligned} \sum_{k=0}^{n-1} |D_k|^2 + \frac{|D_n|^2}{2} &\leq \frac{2n+1}{2} \exp \left\{ \sum_{k=1}^n \left(\frac{k^2}{n(2n+1)} + k - \frac{k^2}{n} \right) |A_k|^2 \right. \\ &\quad \left. - \sum_{k=1}^n \frac{1}{k} - \frac{1}{2n+1} + 1 \right\} \\ &= \frac{2n+1}{2} \exp \left\{ \sum_{k=1}^n \left(\frac{k^2}{n(2n+1)} + k - \frac{k^2}{n} \right) \frac{|p_k - \zeta^k|^2}{k^2} \right. \\ &\quad \left. - \sum_{k=1}^n \frac{1}{k} - \frac{1}{2n+1} + 1 \right\}. \quad (*) \end{aligned}$$

By the corollary to Lemma 1, we can pick a ζ with $|\zeta| = 1$ such that

$$\begin{aligned} \sum_{k=1}^n \left(\frac{k^2}{n(2n+1)} + k - \frac{k^2}{n} \right) \frac{|p_k - \zeta^k|^2}{k^2} \\ \leq \sum_{k=1}^n \left(\frac{k^2}{n(2n+1)} + k - \frac{k^2}{n} \right) \frac{1}{k^2} \\ = \sum_{k=1}^n \left(\frac{1}{n(2n+1)} + \frac{1}{k} - \frac{1}{n} \right) \\ = \frac{1}{2n+1} + \sum_{k=1}^n \frac{1}{k} - 1. \end{aligned}$$

Hence, there exists a ζ with $|\zeta| = 1$ to make the exponent in (*) nonpositive, and we have

$$\sum_{k=0}^{n-1} |D_k|^2 + \frac{|D_n|^2}{2} \leq \frac{2n+1}{2} = n + \frac{1}{2}.$$

This is the statement of Lemma 4.

PROOF OF THE THEOREM. If $f(z)$ is close-to-convex, then $e^{i\alpha}zf'(z) = r(z)g(z)$, where $\operatorname{Re} r(z) > 0$ and

$$r(z) = e^{i\alpha} + \sum_{k=1}^{\infty} r_k z^k.$$

For $|\zeta| = 1$, we have

$$e^{i\alpha} (1 - \zeta z) f'(z) = (1 - \zeta z) \frac{g(z)}{z} r(z),$$

or,

$$\begin{aligned} e^{i\alpha} \left\{ 1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - \zeta n a_n] z^n \right\} \\ = \left[1 + \sum_{k=1}^{\infty} (b_{k+1} - \zeta b_k) z^k \right] \left[e^{i\alpha} + \sum_{k=1}^{\infty} r_k z^k \right]. \end{aligned}$$

Comparing the coefficients of z^n on both sides, we get

$$\begin{aligned} e^{i\alpha} [(n+1)a_{n+1} - \zeta n a_n] &= r_n + r_{n-1}(b_2 - \zeta b_1) + \dots + e^{i\alpha}(b_{n+1} - \zeta b_n) \\ &= \sum_{k=0}^n r_{n-k} D_k, \end{aligned}$$

where $r_0 = e^{i\alpha}$ and $D_k = b_{k+1} - \zeta b_k$. To the first n terms of the right-hand side, we apply the inequality $|ab| \leq |a/2|^2 + |b|^2$, and to the last term the inequality $|b| \leq \frac{1}{2} + |b|^2/2$. We obtain,

$$\begin{aligned}
|(n+1)a_{n+1} - \zeta na_n| &\leq \left| \frac{r_n}{2} \right|^2 + 1 + \left| \frac{r_{n-1}}{2} \right|^2 \\
&\quad + |b_2 - b_1|^2 + \dots + \frac{1}{2} + \frac{1}{2} |b_{n+1} - \zeta b_n|^2 \\
&\leq n + \frac{1}{2} + 1 + \sum_{k=1}^{n-1} |b_{k+1} - b_k|^2 + \frac{1}{2} |b_{n+1} - \zeta b_n|^2,
\end{aligned}$$

where we applied the well-known estimate $|r_k| \leq 2$, for $k = 1, 2, \dots$. By Lemma 4,

$$|(n+1)a_{n+1} - \zeta na_n| \leq n + \frac{1}{2} + n + \frac{1}{2} = 2n + 1,$$

for some ζ with $|\zeta| = 1$. But by the triangle inequality,

$$|(n+1)|a_{n+1}| - n|a_n| \leq |(n+1)a_{n+1} - \zeta na_n|.$$

Hence,

$$|(n+1)|a_{n+1}| - n|a_n| \leq 2n + 1.$$

The general case of inequality follows from this by induction.

In the case of equality, a close inspection of the proof shows that $r(z)$ has the form $(1+tz)/(1-tz)$, with $|t| = 1$. Hence, we have $zf'(z) = ((1+tz)/(1-tz))g(z)$. Multiplying both sides by $(1-tz)$, we get

$$(1-tz) \sum_{n=1}^{\infty} na_n z^n = (1+tz) \sum_{n=1}^{\infty} b_n z^n.$$

Comparing the coefficient of z^{n+1} on both sides, we obtain

$$(n+1)a_{n+1} - tna_n = b_{n+1} + tb_n.$$

Hence,

$$2n+1 = |(n+1)|a_{n+1}| - n|a_n| \leq |b_{n+1} + tb_n|.$$

So our starlike function $g(z)$ must be a rotation of the Koebe function and has the form $z/(1-tz)^2$; hence $f(z)$ is $z/(1-tz)^2$ also.

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