INNER LIMIT DERIVATIONS

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ABSTRACT. We give a condition involving the approximating derivations and a faithful state which, in conjunction with the "core condition", guarantees that the derivation gives rise to an automorphism group of the von Neumann algebra coming from the state via the GNS procedure.

One of the basic problems in the theory of unbounded derivations is to find conditions such that the derivation or an extension gives rise to an automorphism group of the algebra in question. In the von Neumann algebra setting the following was proposed, [2].

"Let $\mathfrak{M}$ have a cyclic and separating vector ξ , and H be a selfadjoint operator giving rise to a densely defined derivation of $\mathfrak{M}$. If $H\xi = 0$ and $\mathfrak{D}(\delta)\xi$ is a core for H , then $\text{ad}(e^{iHt})$ takes $\mathfrak{M}$ into itself".

An elegant solution was given by Bratteli and Haagerup [1]. They showed that the conjecture was indeed true if H and the modular operator Δ_ξ commuted and that this was *necessary*.

For the case of an inner limit derivation acting on a C^* -algebra $\mathfrak{A}$, we formulate a condition involving the approximating derivations and a separating state ω so that if the "core condition" is satisfied in the representation space of ω , we obtain an automorphism group of the von Neumann algebra. The condition is vacuous in case ω is a trace and so we recover the Bratteli-Robinson result [3], with some additional information about the unitary group $t \rightarrow e^{iHt}$.

First we need an easy lemma about the skew information of an operator h relative to a state ω , which as indicated in [4] is "a measure of how far h is from commuting with ω ."

LEMMA. Suppose that $\omega(x) = (x\xi|\xi)$ is a faithful state on a von Neumann algebra $\mathfrak{M}$. Let J be the corresponding modular anti-unitary involution. On the selfadjoint portion of $\mathfrak{M}$ define the seminorms

$$p(h) = \|(I - J)h\xi\|$$

and

$$q(h) = \sup_{\substack{x \in \mathfrak{M} \\ x \neq 0}} |\omega([h, x])| / (\omega(x^*x + xx^*))^{1/2}$$

then $1/\sqrt{2} \ q(h) \leq p(h) \leq q(h)$.

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PROOF. As $\mathfrak{N}\xi$ is a core for $\Delta^{1/2}$, [9],

$$p(h) = \|(I - J)h\xi\| = \sup |((I - \Delta^{1/2})h\xi|(I + \Delta^{1/2})x^*\xi)|$$

where the supremum runs over the dense set of vectors in the unit ball of the form $(I + \Delta^{1/2})x^*\xi$. But

$$((I - \Delta^{1/2})h\xi|(I + \Delta^{1/2})x^*\xi) = -\omega([h, x])$$

so that $p(h) = \sup\{|\omega([h, x])| : \|(I + \Delta^{1/2})x^*\xi\| \leq 1\}$. Noting that

$$\omega(x^*x + xx^*) \leq \|(I + \Delta^{1/2})x^*\xi\|^2 \leq 2\omega(x^*x + xx^*)$$

the proof of the lemma is complete.

Let $\mathfrak{A}$ be a C^* -algebra, δ a densely defined $*$ -derivation on $\mathfrak{A}$ which is inner limit in the sense that there exists a sequence $h_n = h_n^* \in \mathfrak{A}$ such that $\delta(x) = \lim[ih_n, x]$ for all $x \in \mathfrak{D}(\delta)$. (See [8] for $\mathfrak{A}$ hyperfinite and δ a normal $*$ -derivation where this is always satisfied.) Suppose now that ω is a state and $\omega(\delta(x)) = 0$ for all $x \in \mathfrak{D}(\delta)$. If $(\pi_\omega, \mathfrak{H}_\omega, \xi)$ denotes the corresponding GNS triple then the definition

$$iH\pi_\omega(x)\xi = \pi_\omega(\delta(x))\xi, \quad x \in \mathfrak{D}(\delta)$$

gives a densely defined symmetric operator H such that $[iH, \pi_\omega(x)]\eta = \pi_\omega(\delta(x))\eta$ for all $\eta \in \pi_\omega(\mathfrak{D}(\delta))\xi$. The state ω is said to be separating if $y \rightarrow (y\xi|\xi)$ is faithful on $\mathfrak{N} = \pi_\omega(\mathfrak{A})''$.

THEOREM. *Let ω be a separating state on $\mathfrak{A}$. Let δ be an inner limit derivation with $\omega \circ \delta = 0$, and approximating sequence h_n . Suppose that $|\omega([h_n, x])|^2 \leq K\omega(x^*x + xx^*)$. If H is as above and is essentially selfadjoint on $\pi_\omega(\mathfrak{D}(\delta))\xi$, then $\text{ad}(e^{iHt})$ implements an automorphism group of $\mathfrak{N}$.*

PROOF. We show first that $(I - J)\pi_\omega(h_n)\xi$ converges weakly to zero. By the hypothesis and the above Lemma it follows that $(I - J)\pi_\omega(h_n)\xi$ has a subsequence $(I - J)\pi_\omega(h_{n_j})\xi$ converging weakly to some vector $\eta \in \mathfrak{H}$. Let $x^* \in \mathfrak{N}$, such that $(I + \Delta^{1/2})x^*\xi$ approximates η (the set $\mathfrak{N}\xi$ is a core for $\Delta^{1/2}$). We can in fact choose x and j_0 so that for $j \geq j_0$

$$|((I - J)\pi_\omega(h_{n_j})\xi|(I + \Delta^{1/2})x^*\xi)| \geq 1/2$$

but the calculation of the Lemma shows

$$|((I - J)\pi_\omega(h_n)\xi|(I - \Delta^{1/2})x^*\xi)| = |\omega([h_n, x])|.$$

Thus for $j \geq j_0$, $|\omega([h_{n_j}, x])| \geq 1/2$. The inequality $|\omega([h_n, y])|^2 \leq K\{\omega(y^*y) + \omega(yy^*)\}$ extends to all of $\mathfrak{N}$ and allows us to replace x by some $y \in \mathfrak{D}(\delta)$ and conclude that

$$|\omega([h_{n_j}, y])| \geq 1/4 \quad \text{for } j \geq j_0.$$

However, $[ih_{n_j}, y] \rightarrow \delta(y)$ and $\omega(\delta(y)) = 0$, a contradiction. Thus $\eta = 0$. As this holds for all subsequences, $(I - J)\pi_\omega(h_n)\xi \rightarrow 0$, weakly.

Since $(I - J)\pi_\omega(h_n)\xi$ converges weakly to zero we may choose a sub-

sequence, $(I - J)\pi_\omega(h_n)\xi$ which converges $C - 1$ to zero in norm [10]. Relabeling and letting $T_j = 1/j \sum_{n=1}^j h_n$ we have $(I - J)\pi_\omega(T_j)\xi \rightarrow 0$ in norm as $j \rightarrow \infty$. It is a simple matter to verify that $[iT_j, x] \rightarrow \delta(x)$ as $j \rightarrow \infty$.

Define $H_j = \pi_\omega(T_j) - J\pi_\omega(T_j)J$. Then for $x \in \mathfrak{D}(\delta)$ we have

$$\begin{aligned} H_j \pi_\omega(x) \xi &= [H_j, \pi_\omega(x)] \xi + \pi_\omega(x) H_j \xi \\ &= [\pi_\omega(T_j), \pi_\omega(x)] \xi + \pi_\omega(x) (I - J) \pi_\omega(T_j) \xi. \end{aligned}$$

The first term converges to $-i\pi_\omega(\delta(x))\xi = H\pi_\omega(x)\xi$ and the second converges to zero by the above calculations. As $\pi_\omega(\mathfrak{D}(\delta))\xi$ is a core for H , it follows, [5, p. 428], $e^{iH_j t} \rightarrow e^{iHt}$ strongly. Since

$$e^{iH_j t} y e^{-iH_j t} = e^{i\pi_\omega(T_j)t} y e^{-i\pi_\omega(T_j)t}$$

for all $y \in \mathfrak{N}$, it follows that $e^{iHt} \cdot e^{-iHt}$ is an automorphism group of $\mathfrak{N}$.

REMARKS. 1. There is a slight difference between our core condition and that of [2] in that $\mathfrak{D}(\delta)$ there is for the extension of δ to $\mathfrak{N}$.

2. It is straightforward to show that the bound in the skew information of h_n is equivalent to

$$\omega(e^{ih_n t} x e^{-ih_n t}) \leq e^{\gamma t} \omega(x)$$

for $t \geq 0$, $x \geq 0$, and some constant γ . An inequality of the form

$$\omega(e^{ih_n t} x e^{-ih_n t}) \leq f(t) \omega(x)$$

is satisfied in some classical models [6]. Here $f(t)$ is a bounded continuous function on the real line.

3. One way to obtain the inequality on h_n , is related to the modular automorphism for ω . In that case the constant K could be given by the Bogoliubov inequality, [7], as $\sup_n |((\log \Delta) h_n \xi | h_n \xi)|$ provided this number is finite. This involves a certain amount of analyticity for the h_n with respect to the modular automorphism. We note that if this automorphism group preserves $\mathfrak{A}$ and acts points norm continuously then the h_n can be replaced by analytic elements.

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