MEAN BOUNDARY VALUE PROBLEMS FOR A CLASS OF ELLIPTIC EQUATIONS IN E^3

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ABSTRACT. An interior Dirichlet problem for generalized axisymmetric potentials is solved constructively by means of a Riemann series expansion determined from the arithmetic means of the boundary values. The problem is generalized to a class of axisymmetric elliptic partial differential equations in E^3 . The Bergman and Gilbert integral operator method is used along with the method of ascent.

Introduction. Generalized axisymmetric potentials (GASP) are regular solutions $F^\alpha = F^\alpha(x, y)$ of the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{(2\alpha + 1)}{y} \frac{\partial}{\partial y} \right) F^\alpha = 0, \quad \alpha > -\frac{1}{2},$$

in a neighborhood of the origin that are subject to the Cauchy data $\partial F^\alpha / \partial y = 0$, $F^\alpha = f$ along the singular line $y = 0$. They expand uniquely in ultraspherical harmonics,

$$F_n^\alpha(x, y) = r^n P_n^{(\alpha, \alpha)}(\cos \theta) / P_n^{(\alpha, \alpha)}(1), \quad n = 0, 1, 2, \dots,$$

$x = r \cos \theta$, $y = r \sin \theta$, as Fourier series

$$F^\alpha(x, y) = \sum_{n=0}^{\infty} a_n F_n^\alpha(x, y) \tag{1}$$

(the $P_n^{(\alpha, \alpha)}$ are ultraspherical polynomials [1]) which are interpreted as "Poisson" integrals of ultraspherical expansions. If $2\alpha = k - 3$, $k = 3, 4, \dots$, F^α is an axisymmetric potential in E^k .

The Bergman and Gilbert integral operator method [2], [7], [8] is instrumental in determining properties of a GASP F^α such as: location of singularities, zeros, extremal properties and growth from its Fourier coefficients by extending corresponding properties of associated analytic functions. The method of ascent [8] extends some of these properties to solutions Ψ of wider classes of elliptic equations. Generally speaking, as a consequence of ascent, when Ψ and an associated F^α are identical on the Liapunov boundary of a region and a property is satisfied by F^α , it extends to

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Ψ by a maximum principle and/or analytic continuation. This is of special interest in approximation theory where recent developments on the Dirichlet problem concern both approximation of a GASP by harmonic interpolating polynomials [6], [9] and determination of analytical properties of a GASP from its boundary values [12]–[14].

Given an axisymmetric potential $F = F(x, y)$ on an axisymmetric region $\Omega \subset E^3$ and a set Γ_n of $n + 1$ circles in an axisymmetric subregion $A \subset \Omega$, M. Marden [9] utilized the Bergman-Whittaker operator [2, p. 44] to construct axisymmetric harmonic polynomials Λ_n and an axisymmetric region $B \subset \Omega$ so that Λ_n interpolates to F and its partial derivatives with respect to x on A and $\Lambda_n \rightrightarrows F$ on $\text{cl}(B)$. Properties of the Λ_n for coincident interpolating circles were found.

For real GASP F^α regular on the open unit sphere Σ and continuous on $\text{cl}(\Sigma)$, P. A. McCoy [12], [13] showed that Chebyshev approximation over classes of harmonic polynomials gives necessary and sufficient conditions for F^α to be harmonically continued as an entire function. Also shown is that the error in the approximation determines the (infinite, logarithmic) order and type of F^α . These characterizations used operators developed in [10], [11] to extend classical theorems [15], [16] from function theory. The location of the finite singularities of the harmonic continuation of the principal branch of an axisymmetric potential regular on $\text{cl}(\Sigma)$ is determined from the error in the Chebyshev approximation over classes of Newtonian potentials in [14].

This paper solves the interior Dirichlet problem for GASP by constructing a Riemann series GASP expansion from the so-called Riemann series introduced by C. H. Ching and C. K. Chui [5] for GASP expansion from the arithmetic means of the boundary data. The construction uses an invertible operator W_α [12] related to A_μ [7, p. 168] to extend an analogous problem for analytic functions in the disk solved in [4], [5].

Preliminaries. The continuous operator W^α defined by

$$F^\alpha(x, y) = W_\alpha[f] = \int_0^\pi f(x + iy \cos t) d\mu_\alpha(t), \quad (2)$$

$$d\mu_\alpha(t) = \left[\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)/\Gamma(\alpha + 1) \right] (\sin t)^{2\alpha} dt, \quad (3)$$

is for each $\alpha \geq 0$ a one-one map from the space of associated functions f analytic in the disk $\Delta: x^2 + y^2 < 1$ onto the space of GASP F^α regular in the sphere (upper semi-disk) $\Sigma: x^2 + y^2 < 1, y \geq 0$. It is convenient to harmonically continue F^α as an even GASP function element H^α on the disk Δ by reflecting F^α as $F^\alpha(x, y) = F^\alpha(x, -y)$. These functions k , regular (according to context) in a region Ω that extend continuously to $\text{cl}(\Omega)$ are indicated by $k \in \mathfrak{R}(\Omega)$. The boundary function k is the restriction of k to $\partial\Omega$.

Let $\mu(n)$ be the Möbius function and

$$q_n(z) = \sum_{s|n} \mu(s|n) z^s,$$

with $z = x + iy$, be the polynomial basis introduced in [5] for solving mean boundary value problems. Here and throughout, $s|n$ means that s is a factor of n . The harmonic polynomials $H_0^\alpha(x, y) = 1$, and

$$H_n^\alpha(x, y) = \sum_{s|n} \mu(n|s) F_n^\alpha(x, y), \quad n = 1, 2, \dots, \quad (4)$$

which are the image of $q_n(z)$ under W_α , can be extended as even function elements by reflection to form a complete set for regular GASP H^α on Δ relative to the norm $\|\cdot\|_\rho$ defined by $\|q\|_\rho = \sup\{|q(x, y)|: x^2 + y^2 \leq \rho^2\}$. The completeness follows from a result in [5].

The space of all continuous boundary functions h with (trigonometric) Fourier coefficients $a_n(h) = 0$, $n < 0$, for which $a_n(h) = O(1/n^{1+\epsilon})$ (given $\epsilon > 0$) is designated as A_ϵ . The arithmetic means of h are taken at the roots of unity as

$$s_n(h) = \frac{1}{n} \sum_{k=1}^n h(e^{2\pi i k/n}), \quad n = 1, 2, \dots,$$

and

$$s_\infty(h) = \lim_{n \rightarrow \infty} s_n(h) = \int_{|z|=1} h(z) ds$$

where s is arclength. The Riemann coefficients $r_n(h) = s_n(h) - s_\infty(h)$, $n = 1, 2, \dots$, and $r_0(h) = s_\infty(h)$ are fundamental in the solution to the mean boundary value problem. For more information on Riemann coefficients, see [4], [5] and subsequent papers of Ching and Chui on this subject.

The boundary value problem for GASP. The mean boundary value problem is solved on a disk and then extended to more general regions.

THEOREM 1. *Let the even 2π -periodic function H be a member of class A_ϵ for some $\epsilon > 0$. Then the Riemann series expansion*

$$H^\alpha(x, y) = \sum_{n=0}^{\infty} r_n(H) H_n^\alpha(x, y) \quad (5)$$

represents the harmonic extension of H in the disk Δ as a GASP function element. Furthermore, the following bounds hold:

$$\left| H^\alpha(x, y) - \sum_{n=0}^k r_n(H) H_n^\alpha(x, y) \right| \leq C(\delta, H) k^{-\delta} \quad (i)$$

for $k \geq 1$, $\delta < \epsilon$ and $x^2 + y^2 \leq 1$, and

$$\left| H^\alpha(x, y) - \sum_{n=0}^k r_n(H) H_n^\alpha(x, y) \right| \leq \frac{K(H)}{k^\epsilon} (1 - (x^2 + y^2))^{-1} \quad (ii)$$

for $k \geq 1$ and $x^2 + y^2 < 1$.

PROOF. Construct the function h ,

$$h(z) = \sum_{n=0}^{\infty} \sigma_n q_n(z), \quad \sigma_n = r_n(H), \quad n = 0, 1, 2, \dots,$$

with 2π -periodic boundary values representing the analytic extension of H in Δ as in [5]. The transform $H^\alpha = W_\alpha[h]$ is a GASP expansion uniformly and absolutely convergent on compacta of Δ . To verify that H^α has the correct boundary values, note the normalization $W_\alpha[1] = 1$ used in the estimates

$$\left| H^\alpha(x, y) - \sum_{n=0}^k \sigma_n H_n^\alpha(x, y) \right| \leq \left\| h - \sum_{n=0}^k \sigma_n q_n \right\|_\rho, \quad k = 1, 2, \dots, \\ x^2 + y^2 \leq \rho^2. \quad (6)$$

However, $h \in \mathcal{R}(\Delta)$; then [5, p. 235] applies and

$$\left| H^\alpha(x, y) - \sum_{n=0}^k \sigma_n H_n^\alpha(x, y) \right| < \lim_{\rho \uparrow 1} \left\| h - \sum_{n=0}^k \sigma_n q_n \right\|_\rho \\ = \left\| h - \sum_{n=0}^k \sigma_n q_n \right\|_{\rho=1} \leq C(\delta, H) k^{-\delta}. \quad (7)$$

This estimate establishes (i) and letting $k \rightarrow \infty$ verifies that (unique) H^α has the required boundary values. To establish the second bound apply [5, p. 325] to (6),

$$\left| H^\alpha(x, y) - \sum_{n=0}^k \sigma_n H_n^\alpha(x, y) \right| < \left\| h - \sum_{n=0}^k \sigma_n q_n \right\|_{\rho=x^2+y^2} \\ \leq \kappa(H)/k^\epsilon (1 - (x^2 + y^2)). \quad (8)$$

It now follows directly that

COROLLARY 1. *Let GASP F^α expansion (1) be regular in Σ and continuous on $\text{cl}(\Sigma)$. For each $n \geq 2$, let Γ_n designate the union of the antipodal point $x = 1$ and the set of circles on the intersection of $\partial\Sigma$ and the planes $x_k = \cos(2\pi k/n)$, $k = 1, 2, \dots, [n/2]$. If the boundary function F^α is in class A_ϵ and for each $n \geq 2$ the arithmetic means of F^α taken on the set Γ_n vanish, then $F^\alpha \equiv 0$ on $\text{cl}(\Sigma)$.*

We consider next the problem of a nonspherical domain.

COROLLARY 2. *Let Ω be an open simply connected axisymmetric set about the origin whose boundary is the image of Σ under a twice continuously differentiable diffeomorphism D . Let the set of circles $D_n \subset \partial\Omega$ be the image of Γ_n under D and let GASP F^α be regular in Ω , continuous on $\text{cl}(\Omega)$ with $D^{-1}(H^\alpha)$ a member of A_ϵ . If the Riemann coefficients of F^α are evaluated at the arithmetic means of F^α taken on D_n , then*

$$F^\alpha(x, y) = \sum_{n=0}^{\infty} r_n(F^\alpha) H_n^\alpha(x, y)$$

is the regular GASP in Ω with boundary function F^α .

PROOF. Let w be the projection into the z -plane of the intersection of Ω and a meridian plane. Then solve the inverse problem for the W_α -associate on Δ as in the theorem, conformally map $\tau: \Delta \rightarrow w$ and apply W_α . We close with an application of

The method of ascent. A canonical axisymmetric equation in E^3 is

$$T(\Psi) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \frac{\partial}{\partial y} + A(r^2) \left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right\} + C(r^2) \right] \Psi = 0 \quad (9)$$

with entire function coefficients A and C . The operator D_0 [2, p. 68],

$$\Psi(x, y) = D_0[F] = \int_{-1}^{+1} E(r, t) F(x(1-t^2), y(1-t^2)) dt,$$

is a one-one map from the space of regular axisymmetric potentials F ($\alpha = 0$) in Σ onto the space of regular solutions Ψ in Σ . The kernel E is an entire function of r and continuous in t . If Ψ and associate F are in $\mathcal{R}(\Sigma)$ and Ψ satisfies condition L [2, p. 66], $F = \Psi$ pointwise on $\partial\Sigma$. Then under these circumstances the maximum principle gives

$$\|\Psi\|_\rho \leq \nu(E) \|F\|_{\rho=1}, \quad \nu(E) = \max\{|E(r, t)|: 0 \leq r \leq 1, -1 \leq t \leq 1\}.$$

The complete set $\{\Psi_n\}_{n=0}^\infty$ for regular Ψ in Σ relative to $\|\cdot\|_\rho$ is $\Psi_n = D_0[H_n]$,

$$\Psi_n(x, y) = \sum_{s|n} \mu(n|s) r^n J_s^*(r) F_n(x, y), \quad n = 1, 2, \dots,$$

$$\Psi_0(x, y) \equiv 1, \quad J_n^*(r) = D_0[(1-t^2)^n], \quad J_n^*(1) = 1, \quad n = 0, 2, \dots \quad (10)$$

Reasoning as in Theorem 1 with estimate (7) shows

THEOREM 2. Let the continuous axisymmetric function Ψ with domain $\partial\Sigma$ be in class A_ϵ (given $\epsilon > 0$) and be uniformly expandable in a Fourier-Legendre series. If the Riemann coefficients of Ψ are evaluated from the arithmetic means of Ψ on the set Γ_n , then

$$\Psi(x, y) = \sum_{n=0}^{\infty} r_n(\Psi) \Psi_n(x, y)$$

represents the regular solution of $T(\Psi) = 0$ in Σ with boundary values Ψ . Furthermore, the following bounds hold:

(i) $|\Psi(x, y) - \sum_{n=0}^k r_n(\Psi) \Psi_n(x, y)| \leq \nu(E) C(\delta, \Psi) k^{-\delta}$ for $k \geq 1$, $\delta < \epsilon$ and $x^2 + y^2 \leq 1$, and

(ii) $|\Psi(x, y) - \sum_{n=0}^k r_n(\Psi) \Psi_n(x, y)| \leq \nu(E) K(\Psi) / (k^\epsilon (1 - (x^2 + y^2)))$ for $k \geq 1$ and $x^2 + y^2 < 1$.

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