AN INEQUALITY FOR DOUBLE TANGENTS

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ABSTRACT. For a regular closed curve on the plane it is known that $E = I + X + \frac{1}{2}F$ where E , I , X and F are the numbers of external double tangents, internal double tangents, self-intersections, and inflexion points respectively. It is proven here that if $F = 0$ then I is even and $I < (2X + 1)(X - 1)$. Furthermore, examples are given which show that if the four tuple (E, I, X, F) of nonnegative integers satisfies (a) F even, (b) $E = I + X + \frac{1}{2}F$, and (c) if $F = 0$ then I is even and $I < X(X - 1)$, then there is a regular closed plane curve which realizes these values.

1. Introduction. Let γ be a closed plane curve. Under certain regularity conditions (given below) there are a finite number $E(I)$ of straight lines which are tangent to γ at two points s and t and such that the unit normals at s and t are equal (negatives of each other). These are called the external (internal) double tangents of γ . Also, there will be a finite number X of self-intersections of γ , and a finite number F of inflexion points. It has been shown in Fabricius-Bjerre [F] (see also Halpern [H1], [H2], and Banchoff [B]) $E = I + X + \frac{1}{2}F$.

In this note we will show by Example 1 that if e , i , x and f are nonnegative integers satisfying (a) f is even, (b) $f \geq 2$, and (c) $e = i + x + \frac{1}{2}f$, then there is a curve γ satisfying our regularity conditions such that $(E, I, X, F) = (e, i, x, f)$. Also, we will show by Example 2 that if e , i , and x are nonnegative integers satisfying (1) i is even, (2) $i \leq x(x - 1)$, and (3) $e = i + x$, then there is a curve γ satisfying our regularity conditions such that $(E, I, X, F) = (e, i, x, 0)$. Finally, we will prove in Theorem 1 that if γ satisfies our regularity conditions and $F = 0$, then I is even and $I < (2X + 1)(X - 1)$. There remains a gap between the bound $(2X + 1)(X - 1)$ on I given by Theorem 1 and the values obtained in Example 2, $I \leq X(X - 1)$. We conjecture that if γ satisfies our regularity conditions and $F = 0$, then $I \leq X(X - 1)$. These results are essentially those announced in [H1].

2. Definitions and notations. Let S^1 be the unit circle about the origin in the plane $\mathbb{R}^2$. We suppose γ is given by a C^4 immersion $\gamma: S^1 \rightarrow \mathbb{R}^2$. For each $s \in S^1$, let $\kappa(s)$, $n(s)$, T_s be the curvature, unit normal when $\kappa(s) \neq 0$, and tangent line at s , respectively.

Received by the editors May 31, 1978.

AMS (MOS) subject classifications (1970). Primary 53A05.

Key words and phrases. Double tangent, self-intersection, inflexion point, tangent winding number.

An unordered pair $\{s, t\} \subset S^1$ is a double tangent provided $s \neq t$ and $T_s = T_t$. It is regular if $\kappa(s) \neq 0$ and $\kappa(t) \neq 0$. A regular double tangent $\{s, t\}$ is external (internal) provided $n(s) = n(t)$ ($n(s) = -n(t)$).

A self-intersection is an unordered pair $\{s, t\} \subset S^1$ such that $s \neq t$ and $\gamma(s) = \gamma(t)$. It is regular if $T_s \neq T_t$.

A point $s \in S^1$ is an inflexion point provided $\kappa(s) = 0$. It is regular if $\kappa'(s) \neq 0$.

The curve γ is regular provided all its double tangents, self-intersections, and inflexion points are regular. We denote the numbers of external double tangents, internal double tangents, self-intersections, and inflexion points by E, I, X and F , respectively.

3. An inequality. It is shown in Fabricius-Bjerre [F] and Halpern [H1] that if X is regular, then E, I, X and F are finite and satisfy $E = I + X + \frac{1}{2}F$. We prove here further restrictions on (E, I, X, F) .

THEOREM 1. *If X is regular and $F = 0$, then I is even and $I \leq (2X + 1)(X - 1)$.*

The reader may find it helpful to look at the examples in §4 before reading the proof.

PROOF. Let $s_0 \in S^1$. Because the curvature $\kappa(s)$ never vanishes, the angle from $\gamma'(s_0)$ to $\gamma'(s)$ varies monotonically as s goes around S^1 , starting and ending at s_0 . Without loss of generality we may assume that this angle is monotonically increasing. Thus we can use it to reparameterize γ . Let $\alpha: [0, 2\pi W] \rightarrow \mathbf{R}^2$ be such a reparameterization, where $2\pi W$ is the total angle through which $\gamma'(s)$ turns. The integer W is called the tangent winding number. There is a unique extension of α to a periodic map of the real line $\mathbf{R}$ into $\mathbf{R}^2$ of period $2\pi W$. We let α denote this extension also. We denote the unit tangent vector to α at $s \in \mathbf{R}$ by

$$\dot{\alpha}(s) = \alpha'(t) / \|\alpha'(t)\|.$$

The unit normal is again denoted by $n(s)$.

Because the angle from $\dot{\alpha}(s)$ to $\dot{\alpha}(t)$ is $t - s$, we see that $\dot{\alpha}(s) = \dot{\alpha}(t)$ ($\dot{\alpha}(s) = -\dot{\alpha}(t)$) iff $t = s + j\pi$ for some even (odd) integer j . Thus the pair $\{s, t\} \subset \mathbf{R}$ corresponds to a double tangent iff $t = s + j\pi$ for some integer j which is not a multiple of $2W$ and $\dot{\alpha}(s) \parallel (\alpha(t) - \alpha(s))$ where " $\parallel$ " denotes "parallel to".

In order to tell when such an $\{s, t\}$ corresponds to an external (internal) double tangent, we observe that the angle from $\dot{\alpha}(0)$ to $\dot{\alpha}(s)$, ($= s$), is an increasing function and so $(\dot{\alpha}(s), n(s))$ is a right-handed frame, i.e., it has the same orientation as the standard frame $((1, 0), (0, 1))$. Using the right-handedness of $(\dot{\alpha}(s), n(s))$ and $(\dot{\alpha}(t), n(t))$ we see that $\{s, t\}$ corresponds to an external (internal) double tangent iff j is even (odd).

Let j be an odd integer. Set

$$A_j = \{s \in \mathbf{R} \mid \dot{\alpha}(s) = u(\alpha(s + \pi j) - \alpha(s)) \text{ for some } u > 0\},$$

$$S_j = \{s \in \mathbf{R} \mid \dot{\alpha}(s) = u(\alpha(s + \pi j) - \alpha(s)) \text{ for some } u < 0\}.$$

If $s \in A_j$ ($s \in S_j$), then the portions of α near s and $s + \pi j$ are approaching (separating from) each other.

Claim 1. The points of A_j and S_j alternate, i.e., between successive members of A_j (S_j) there is a member of S_j (A_j).

PROOF. Note that the regularity condition for self-intersections insures that $\alpha(s + \pi j) \neq \alpha(s)$, for all $s \in \mathbf{R}$ (j is odd and hence not a multiple of $2W$). Let $\theta(s)$ be the angle from $\alpha(s + \pi j) - \alpha(s)$ to $\dot{\alpha}(s)$. $\theta(s)$ is uniquely determined if we require $s \rightarrow \theta(s)$ to be continuous and $0 \leq \theta(0) < 2\pi$. It is geometrically clear and easy to verify analytically that if $\theta(s) = \pi k$ for some integer k , then $\theta'(s) > 0$. It follows that the points of $A_j = \theta^{-1}\{0, \pm 2\pi, \pm 4\pi, \dots\}$ and $S_j = \theta^{-1}\{\pm \pi, \pm 3\pi, \pm 5\pi, \dots\}$ alternate.

Claim 2. If $s, s' \in A_j$ and $s \neq s'$, then $|s - s'| > \pi$.

PROOF. Suppose $s \in A_j$. We may suppose that $\alpha(s)$ and $\alpha(s + \pi j)$ lie on the x -axis and that $n(s) = (0, 1)$. Then

$$\dot{\alpha}(s) = (1, 0) = -\dot{\alpha}(s + \pi j).$$

Let $P_y: \mathbf{R}^2 \rightarrow \mathbf{R}$ be the projection onto the y -axis, $P_y(x, y) = y$ for $(x, y) \in \mathbf{R}^2$. Then for $t \in (s, s + \pi)$, $P_y(\alpha'(t)) > 0$ and $P_y(\alpha'(t + \pi j)) < 0$. Hence, for $t \in (s, s + \pi]$, $P_y(\alpha(t)) > 0$, $P_y(\alpha(t + \pi j)) < 0$, and $P_y(\dot{\alpha}(t)) \geq 0$. Hence $\theta(t)/2\pi$ is not an integer and so $t \notin A_j$ for $t \in (s, s + \pi]$. This proves Claim 2.

Claim 3. $A_1 = \emptyset = A_{2W-1}$.

PROOF. Suppose $j = 1$ and $s \in \mathbf{R}$. We may assume $\alpha(s) = 0$, $\dot{\alpha}(s) = (1, 0)$ and $n(s) = (0, 1)$. Then $P_y(\alpha'(t)) > 0$ for $t \in (s, s + \pi)$, and hence

$$P_y(\alpha(s + \pi)) > P_y(\alpha(s)).$$

It follows that $\theta(s)/2\pi$ is not an integer, and so $s \notin A_1$. Since $\alpha(s) = \alpha(s + 2\pi W)$ for all $s \in \mathbf{R}$, a similar argument shows that $A_{2W-1} = \emptyset$. This proves Claim 3.

Let $\#B$ denote the cardinality of B . Set $A'_j = A_j \cap [0, 2\pi W)$, and $S'_j = S_j \cap [0, 2\pi W)$. It is easy to see that

$$2I = \sum_{\substack{1 \leq j \leq 2W-1 \\ j \text{ odd}}} \#A'_j + \#S'_j.$$

From Claim 1 and the periodicity of α we see that for each j , $\#A'_j = \#S'_j$. From Claim 2 we see that for each j , $\#A'_j \leq 2W - 1$. Using Claim 3, we can conclude that $I \leq (W - 2)(2W - 1)$. A well-known theorem of Whitney is that $W \leq X + 1$. Hence $I \leq (2X + 1)(X - 1)$.

We say that an internal double tangent $\{s, t\} \subset S^1$ is approaching

(separating) provided $\gamma'(s) = u(\gamma(t) - \gamma(s))$ for some $u > 0$ ($u < 0$). Let I_A and I_S be the numbers of approaching and separating internal double tangents respectively. It is easy to see that

$$2I_A = \sum_{\substack{1 < j < 2W-1 \\ j \text{ odd}}} \#A'_j \quad \text{and} \quad 2I_S = \sum_{\substack{1 < j < 2W-1 \\ j \text{ odd}}} \#S'_j.$$

Hence $I_A = I_S$, and so $I = I_A + I_S$ is even. Q.E.D.

4. Examples.

EXAMPLE 1. Given nonnegative integers e, i, x and f satisfying (a) f is even, (b) $f \geq 2$, and (c) $e = i + x + \frac{1}{2}f$. We will construct a regular curve γ such that

$$(E, I, X, F) = (e, i, x, f).$$

First we give a construction (Construction A) for a regular curve γ with $F = 2$, $X = 0$ and $I \geq 0$ arbitrary.

This construction is illustrated in the following figures for the cases I even and I odd.

CONSTRUCTION A

Next, we give a construction (Construction B) for a regular curve γ with $I = 0$, $X = 0$, and $F/2$ an arbitrary nonnegative integer. In the following figure illustrating Construction B note the position of the lower wiggles of γ with respect to the tangent lines drawn at the higher inflection points.

Thirdly, we give a method (Construction C) for obtaining extra self-intersections without affecting F or I . This can be done by adjusting two nearly parallel close arcs as illustrated in the figure. This can be done keeping the curvature positive everywhere on these arcs.

Now starting from Construction A with $I = i$, we adjust it by Constructions B and C to obtain Example 1 illustrated below.

CONSTRUCTION B

CONSTRUCTION C BEFORE

CONSTRUCTION C AFTER

EXAMPLE 1

EXAMPLE 2. Given nonnegative integers e , i and x satisfying (1) i is even, (2) $i \leq x(x-1)$, and (3) $e = i + x$. We will construct a regular curve γ such that

$$(E, I, X, F) = (e, i, x, 0).$$

Since $i \leq x(x-1)$ we have $i/2 \leq \binom{x}{2}$ = combinations of x things taken 2 at a time. Hence we can find an n , such that $0 \leq n \leq x$ and $\binom{n-1}{2} \leq i/2 \leq \binom{n}{2}$.

EXAMPLE 2

Set $k = \binom{n}{2} - I/2$. Then $0 \leq k \leq n-1$. Set $u = x - n$. In the figure below $n = 8$, $k = 3$ and $u = 2$. Consequently $I = i = 2(\binom{8}{2} - 3) = 50$, $X = x = n + u = 10$, and $E = I + X = i + x = e = 60$. The numbers n , k and u guide the construction of Example 2 as follows. The number of small and tiny loops is n . The number of tiny loops inside the top small loop is k . The number of big loops is u .

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