A COMBINATORIAL FORMULA FOR NORMAL STIEFEL-WHITNEY CLASSES¹

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ABSTRACT. Using an ordering of the vertices of a combinatorial n -manifold K , we give an explicit description of a simplicial mod 2 cycle $c_{n-i}(K)$ which represents the dual of the i th normal Stiefel-Whitney class of K .

1. The formula. A *local ordering* of a simplicial complex K is a partial ordering of the vertices of K which restricts to a total order on the closed star of each vertex. If K is a locally ordered n -dimensional complex, and (σ, τ) is an ordered pair of n -simplices of K such that $\sigma \cap \tau \neq \emptyset$, define $\mu(\sigma, \tau) \in \mathbb{Z}/2\mathbb{Z}$ as follows. Since $\sigma \cap \tau \neq \emptyset$, there is a vertex v such that σ and τ are in $\text{star}(v)$. Let $v_1, \dots, v_s$ be the vertices of σ and τ in the given total order of $\text{star}(v)$. (If a vertex belongs to both σ and τ , it should be listed only once.) Let $\mu(\sigma, \tau) = 1$ if $v_1, v_3, v_5, \dots$ are in σ and $v_2, v_4, v_6, \dots$ are in τ , and $\mu(\sigma, \tau) = 0$ otherwise. For $i = 1, 2, \dots, n$, let $c_{n-i}(K)$ be the simplicial mod 2 $(n-i)$ -chain $\sum \mu(\sigma, \tau) \sigma \cap \tau$, summed over all ordered pairs (σ, τ) of n -simplices such that $\dim(\sigma \cap \tau) = n - i$.

THEOREM 1. *Let K be a finite locally ordered combinatorial n -manifold without boundary. The chain $c_{n-i}(K)$ is a mod 2 cycle, and its homology class is Poincaré dual to $\bar{w}_i(K)$, the i th normal Stiefel-Whitney class of K .*

A theorem of Levitt and Rourke [6] asserts that some combinatorial formula using a local ordering exists for every characteristic class of combinatorial manifolds. The prototype is the combinatorial formula for the tangential Stiefel-Whitney classes w_i (cf. [4]). A generalization of this formula for w_i has been given by Goldstein and Turner [3]. The observation that the Goldstein-Turner formula can be proved using our geometric definition of Stiefel-Whitney classes ([2], [7]) led us to the formula for $\bar{w}_i$ presented here. Its form was simplified by a remark of Lee Rudolph.

2. An example. Let K be the triangulation of the projective plane obtained by identifying opposite points of the icosahedron. Order the vertices of K as in Figure 1.

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FIGURE 1

The cycle $c_1(K)$ is the sum of the nine edges emphasized in the figure. This cycle represents the nonzero class of $H_1(K)$ (mod 2 coefficients). For example, consider the edge $\langle 1, 5 \rangle$. The two 2-simplices incident to it are $\langle 1, 4, 5 \rangle$ and $\langle 1, 3, 5 \rangle$. Since $\mu(\langle 1, 4, 5 \rangle, \langle 1, 3, 5 \rangle) = 1$, the coefficient of $\langle 1, 5 \rangle$ in $c_1(K)$ is 1.

The cycle $c_0(K)$ is the sum of the vertices $\langle 3 \rangle$ and $\langle 6 \rangle$, so its homology class is 0. For example $\mu(\langle 1, 3, 5 \rangle, \langle 1, 2, 4 \rangle) = 1$, $\mu(\langle 1, 3, 6 \rangle, \langle 1, 2, 4 \rangle) = 1$, and $\mu(\sigma, \tau) = 0$ for all other ordered pairs of 2-simplices (σ, τ) in the star of $\langle 1 \rangle$. Therefore the coefficient of $\langle 1 \rangle$ in $c_0(K)$ is 0.

Now let $f: K \rightarrow R^2$ be a simplexwise-linear map which puts the vertices of K in order along the parabola (t, t^2) . The cycle $c_1(K)$ is equal to the "fold cycle" of f . Figure 2 shows the image of the star of the edge $\langle 1, 5 \rangle$ under such a map.

FIGURE 2

Similarly, let $g: K \rightarrow R^3$ be a simplexwise-linear map which puts the vertices of K in order along the twisted cubic (t, t^2, t^3) . The cycle $c_0(K)$ is equal to the "branch cycle" of g . In other words, if σ and τ are 2-simplices such that $\sigma \cap \tau$ is a vertex, then

$$g(\sigma^\circ) \cap g(\tau^\circ) \neq \emptyset$$

if and only if either $\mu(\sigma, \tau) = 1$ or $\mu(\tau, \sigma) = 1$. (σ° denotes the interior of σ .) Figure 3 shows the image of the union of $\langle 1, 3, 5 \rangle$ and $\langle 1, 2, 4 \rangle$ under such a map.

FIGURE 3

We will prove Theorem 1 by showing that, for any locally ordered n -complex K , the chain $c_{n-i}(K)$ is the branch chain of some map from K to R^{n+i-1} .

3. The branch cycle. If $f: X \rightarrow Y$ is a continuous map, let $B(f)$ be the set of points $x \in X$ such that for every neighborhood U of x there exist y and z in U with $y \neq z$ and $f(y) = f(z)$. The closed set $B(f)$ is the *branch locus* of f . The branch locus of f is empty if and only if f is locally injective. For example, if $f: N^n \rightarrow P^p$ is a stable differentiable map of smooth manifolds, with $n < p$, then $B(f)$ is precisely the singularity set of f (cf. [7], [1]).

Let $f: K^n \rightarrow R^p$ be a simplexwise-linear map of the n -complex K to Euclidean p -space, $n \leq p$. The map f is *general* if $\dim f(\sigma) = \dim \sigma$ for all simplices σ of K , and

$$\dim(f(\sigma^\circ) \cap f(\tau^\circ)) = \dim \sigma + \dim \tau - p$$

for all pairs of distinct simplices σ, τ of K with $f(\sigma^\circ) \cap f(\tau^\circ) \neq \emptyset$. (No restriction is made on the dimension of intersection of three or more sim-

plices.) If the span of $\{f(v_0), \dots, f(v_p)\}$ is R^p for every set of $p + 1$ distinct vertices $v_0, \dots, v_p$ of K , then f is general.

If $f: K^n \rightarrow R^p$ is a general simplexwise-linear map, $n \leq p$, a simplicial mod 2 $(2n - p - 1)$ -chain $\mathfrak{B}(f)$ in K , with support contained in $B(f)$, is defined as follows. The coefficient of the $(2n - p - 1)$ -simplex ω in the chain $\mathfrak{B}(f)$ is the number (mod 2) of unordered pairs of n -simplices $\{\sigma, \tau\}$ in the star of ω such that $f(\sigma^\circ) \cap f(\tau^\circ) \neq \emptyset$ (cf. Figures 2 and 3). The chain $\mathfrak{B}(f)$ is the *branch chain* of f .

THEOREM 2. *Let $f: K^n \rightarrow R^{n+i-1}$ be a general simplexwise-linear map of the finite combinatorial n -manifold K^n to Euclidean $(n + i - 1)$ -space, $i \geq 1$. The branch chain $\mathfrak{B}(f)$ is a mod 2 cycle, and its homology class is Poincaré dual to $\bar{w}_i(K)$.*

This theorem was conjectured in [7] and proved in [8], [9], and [5], in increasingly general settings. Here is a summary of the proof. If K is a finite purely n -dimensional complex with mod 2 boundary ∂K , and $f: K \rightarrow R^{n+i-1}$ ($i \geq 1$) is general, then $\partial \mathfrak{B}(f)$ lies in ∂K , and the class of $\mathfrak{B}(f)$ in $H_{n-i}(K, \partial K)$ is independent of f . Let $\mathfrak{B}^i(K)$ be this class. The function which assigns the class $\mathfrak{B}^i(K)$ to the complex K is a *geometric homology operation* of degree i which is uniquely characterized by a self-intersection property (cf. [9], [5]). Therefore $\mathfrak{B}^i(K) = \text{Sq}^i[K]$, where Sq^i is the i th dual Steenrod homology operation and $[K] \in H_n(K, \partial K)$ is the fundamental class. Thom's formula for Stiefel-Whitney classes implies that Sq^i of the fundamental class of a manifold is Poincaré dual to $\bar{w}_i$.

To prove Theorem 1 it is enough to show that a local ordering of the vertices of an n -complex K^n can be used to define a general map $f: K^n \rightarrow R^{n+i-1}$ such that $c_{n-i}(K) = \mathfrak{B}(f)$.

Let K be an n -complex. Given a local ordering of the vertices of K and an integer $p \geq n$, a *moment map* $f: K^n \rightarrow R^p$ is a simplexwise-linear map which puts the vertices of K in order along the moment curve $(t, t^2, \dots, t^p)$. That is, if v_i is a vertex of K , then

$$f(v_i) = (t_i, t_i^2, \dots, t_i^p),$$

where $t_i < t_j$ if $v_i < v_j$ in the given ordering.

LEMMA 1. *Any moment map is general.*

PROOF. If $f: K^n \rightarrow R^p$ is a moment map and $v_0, \dots, v_p$ are distinct vertices of K , then the span of $\{f(v_0), \dots, f(v_p)\}$ is R^p if and only if the matrix

$$\begin{bmatrix} 1 & t_0 & t_0^2 & \dots & t_0^p \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_p & t_p^2 & \dots & t_p^p \end{bmatrix}$$

has rank p . But the determinant of this matrix is $\prod_{i > j} (t_i - t_j) \neq 0$.

LEMMA 2 (A. SHAPIRO [10, p. 260]). *Let A and B be simplices in R^p with distinct vertices lying on the moment curve C and $\dim A + \dim B = p$. Then $A \cap B \neq \emptyset$ if and only if the vertices of A and B alternate along C .*

Note that the alternation of vertices implies that $|\dim A - \dim B| \leq 1$.

LEMMA 3. *Let $f: K^n \rightarrow R^p$ be a general simplexwise-linear map, and let σ, τ be n -simplices of K such that $\omega = \sigma \cap \tau$ is a $(2n - p - 1)$ -simplex. Then $f(\sigma^\circ) \cap f(\tau^\circ) \neq \emptyset$ if and only if there are faces $\sigma' < \sigma$ and $\tau' < \tau$ such that $\sigma' \cap \tau' = \emptyset$, $\dim \sigma' + \dim \tau' = p$, and*

$$f((\sigma')^\circ) \cap f((\tau')^\circ) \neq \emptyset.$$

Moreover, σ' and τ' are unique, and $f(\sigma) \cap f(\tau)$ is the cone over $f(\omega)$ from a , where

$$\{a\} = f((\sigma')^\circ) \cap f((\tau')^\circ)$$

(cf. Figures 2 and 3).

PROOF. For any simplex ρ of K , let $v(\rho)$ be the set of vertices of ρ . If σ' and τ' have the stated properties, then $v(\sigma') \cup v(\tau') = v(\sigma) \cup v(\tau)$. For $v(\sigma') \cup v(\tau') \subset v(\sigma) \cup v(\tau)$, and both these sets have $p + 2$ elements. Therefore $v(\omega) \cup v(\sigma') = v(\sigma)$, since $v(\omega) \cup v(\sigma') \subset v(\sigma) \subset v(\sigma) \cap (v(\sigma') \cup v(\tau')) = (v(\sigma) \cap v(\tau')) \cup v(\sigma') \subset v(\omega) \cup v(\sigma')$. Similarly $v(\omega) \cup v(\tau') = v(\tau)$. Now let $a_1 \in (\sigma')^\circ$ and $a_2 \in (\tau')^\circ$ with $f(a_1) = f(a_2) = a \in f((\sigma')^\circ) \cap f((\tau')^\circ)$, and let $b \in \omega^\circ$. Then $(a_1 + b)/2 \in \sigma^\circ$ since σ' and ω span σ . Similarly $(a_2 + b)/2 \in \tau^\circ$. So $f((a_1 + b)/2) = f((a_2 + b)/2) = (a + f(b))/2 \in f(\sigma^\circ) \cap f(\tau^\circ)$, and $f(\sigma^\circ) \cap f(\tau^\circ) \neq \emptyset$. (Note that the cone from a over $f(\omega)$ is contained in $f(\sigma) \cap f(\tau)$.)

Conversely, suppose that $f(\sigma^\circ) \cap f(\tau^\circ) \neq \emptyset$. Now $f(\sigma) \cap f(\tau)$ is a convex cell in R^p containing $f(\omega)$. Furthermore $f(\sigma) \cap f(\tau)$ is the union of the cells of a complex C . The open cells of C are the sets $f(s^\circ) \cap f(t^\circ)$ for $s < \sigma$ and $t < \tau$. (Clearly $f(\sigma) \cap f(\tau)$ is the disjoint union of the open cells $f(s^\circ) \cap f(t^\circ)$, and the boundary of the closed cell $f(s) \cap f(t)$ is the union of $f((s')^\circ) \cap f((t')^\circ)$ for $s' < s$ and $t' < t$, $(s', t') \neq (s, t)$.) The open cells of C which are contained in $f(\omega)$ are the cells $f(w^\circ)$ with $w < \omega$, i.e. the cells $f(s^\circ) \cap f(t^\circ)$ with $s = t$.

Now suppose $s < \sigma$ and $t < \tau$ are such that $s \neq t$ and $\dim(f(s^\circ) \cap f(t^\circ)) > 1$. Since f is general, this implies $\dim s + \dim t \geq p + 1$. Since $v(\sigma) \cup v(\tau)$ has $p + 2$ elements, it follows that $s \cap t \neq \emptyset$. Thus the cell $f(s) \cap f(t)$ has some face in the cell $f(\omega)$. So each edge of the complex C has at least one vertex in $f(\omega)$. Therefore C has a unique vertex $a \notin f(\omega)$, and the convex cell $f(\sigma) \cap f(\tau)$ is the cone from a over $f(\omega)$. Let $\{a\} = f((\sigma')^\circ) \cap f((\tau')^\circ)$ for $\sigma' < \sigma$ and $\tau' < \tau$. Since f is general, $\dim \sigma' + \dim \tau' = p$, so $v(\sigma') \cup v(\tau') = v(\sigma) \cup v(\tau)$ and $\sigma' \cap \tau' = \emptyset$. This completes the proof.

PROPOSITION. Let K be a locally ordered n -complex, and let $f: K \rightarrow R^p$ be a moment map. Then

$$\mathfrak{B}(f) = c_{2n-p-1}(K).$$

PROOF. Let ω be a $(2n - p - 1)$ -simplex of K , and let σ and τ be n -simplices in the star of ω . We must show that $f(\sigma^\circ) \cap f(\tau^\circ) \neq \emptyset$ if and only if $\sigma \cap \tau = \omega$ and either $\mu(\sigma, \tau) = 1$ or $\mu(\tau, \sigma) = 1$. Let $v_1, \dots, v_s$ be the vertices of σ and τ in the given order.

If $\sigma \cap \tau = \omega$ then each of σ and τ has $n + 1$ vertices, $2n - p$ of which are repeated, so $s = p + 2$. If $\mu(\sigma, \tau) = 1$ we consider the two simplices $\sigma' = \langle v_1, v_3, v_5, \dots \rangle$, $\tau' = \langle v_2, v_4, v_6, \dots \rangle$. The simplex σ' is a face of σ and τ' is a face of τ . These simplices are disjoint and $\dim \sigma' + \dim \tau' = p$. By Shapiro's lemma, $f(\sigma') \cap f(\tau') \neq \emptyset$, so Lemmas 1 and 3 imply that $f(\sigma^\circ) \cap f(\tau^\circ) \neq \emptyset$. We obtain the same conclusion if $\mu(\tau, \sigma) = 1$.

Conversely, suppose that $f(\sigma^\circ) \cap f(\tau^\circ) \neq \emptyset$. By Lemma 1,

$$\dim f(\sigma^\circ) \cap f(\tau^\circ) = 2n - p$$

and since $\sigma \cap \tau \supset \omega$, we have

$$\begin{aligned} 2n - p &= \dim(f(\sigma) \cap f(\tau)) > \dim f(\sigma \cap \tau) \\ &\geq \dim \omega = 2n - p - 1. \end{aligned}$$

It follows that $\dim f(\sigma \cap \tau) = \dim \omega$ so $\sigma \cap \tau = \omega$. So Lemmas 1 and 3 imply that there are faces $\sigma' < \sigma$ and $\tau' < \tau$ with $\sigma' \cap \tau' = \emptyset$, $\dim \sigma' + \dim \tau' = p$, and $f((\sigma')^\circ) \cap f((\tau')^\circ) \neq \emptyset$. By Shapiro's lemma, the vertices of σ' and τ' alternate along the moment curve, so $\mu(\sigma, \tau) = 1$ or $\mu(\tau, \sigma) = 1$.

This Proposition and Theorem 2 imply Theorem 1.

4. Questions. The proposition and the proof of Theorem 2 show that if K is any finite purely n -dimensional complex then $c_{n-i}(K)$ is a cycle which represents $\text{Sq}^i[K] \in H_{n-i}(K, \partial K)$. Our combinatorial definition of $c_{n-i}(K)$ is very similar to Steenrod's original definition of Sq^i in cohomology using cup- i products [11]. Can Theorem 1 be proved directly using this relationship?

Our formula for the normal class $\bar{w}$ is also similar to Goldstein and Turner's formula [3] for the tangential class w . Can this relation be used to give a combinatorial proof of the Whitney duality theorem $w \cup \bar{w} = 1$, or something more general?

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