SOME EXTENSIONS OF THE BROCK-CARLITZ IDENTITY

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ABSTRACT. The paper deals with a number of generalizations of an identity first introduced by Brock, and subsequently investigated in some detail by Carlitz. The theorem presented in this paper not only extends the formulas of Carlitz, but also shows that a multi-dimensional function satisfies the identity. A result is also deduced regarding the positive nature of the recurrence relation.

I. Introduction. Brock proposed the identity

$$H(m, n) - H(m-1, n) - H(m, n-1) = \binom{m+n}{n}^2 \quad (1.1)$$

for

$$H(m, n) = \sum_{i=0}^m \sum_{j=0}^n \binom{i+j}{i} \binom{m-i+j}{m-i} \binom{i+n-j}{n-j} \binom{m-i+n-j}{m-i}.$$

The identity was subsequently proved by Slepian [1] and a second combinatorial proof was given by Baer and Brock [2]. Over a period of time, Carlitz [3]–[10] investigated (1.1) in many different directions, including multiple cyclic sums, generating functions, and higher order identities. Interest in (1.1) is heightened by the fact that it is connected with generating functions involving the Jacobi polynomial (see Carlitz [8], also Singhal [12] and Cohen [11]).

One of the papers by Carlitz [7] treats as a main result the expression

$$\begin{aligned} &H(m, n|r, s) \\ &= \sum_{i=0}^{\min(m,r)} \sum_{j=0}^{\min(n,s)} \binom{i+j}{i} \binom{m-i+j}{m-i} \binom{i+n-j}{n-j} \binom{r-i+s-j}{r-i} \end{aligned}$$

which is shown to satisfy the relation

$$\begin{aligned} &H(m, n|r, s) - H(m-1, n|r-1, s) - H(m, n-1|r, s-1) \\ &= \binom{m+n}{m} \binom{r+s}{r} \end{aligned} \quad (1.2)$$

for $m \leq r$ or $n \leq s$. Clearly, by letting $m = r$, $n = s$, (1.1) presents itself.

A special case of our theorem in this paper yields an extension of (1.2).

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Defining

$$H^\beta(m, n|r, s) = \sum_{j=0}^{\min(n,s)} \sum_{i=0}^{\min(m,r)} \binom{i+j}{i} \binom{m-i+j}{m-i} \cdot \binom{\beta+n-j+i}{n-j} \binom{r+s-i-j}{r-i} \quad (1.3)$$

then

$$H^\beta(m, n|r, s) - H^\beta(m-1, n|r-1, s) - H^\beta(m, n-1|r, s-1) = \binom{r+s}{s} \binom{\beta+m+n}{\beta+m} \quad (1.4)$$

provided $m \leq r$ or $n \leq s$. $\beta = 0$ in (1.4) gives (1.2). In point of fact our theorem gives a more general expression for

$$H^{\beta,\gamma}(m, n|r, s) = \sum_{j=0}^{\min(n,s)} \sum_{i=0}^{\min(m,r)} \binom{\gamma+i+j}{i} \binom{m-i+j}{m-i} \cdot \binom{\beta+n-j+i}{n-j} \binom{r+s-i-j}{r-i} \quad (1.5)$$

$$H^{\beta,\gamma}(m, n|r, s) - H^{\beta,\gamma}(m-1, n|r-1, s) - H^{\beta,\gamma}(m, n-1|r, s-1) = \binom{\beta+m+n}{\beta+m} \sum_{k=0}^{\min(m,r)} \binom{r+s-k}{s} \binom{\gamma+k-1}{k} \quad (1.6)$$

provided $m \leq r$ or $n \leq s$. For the restriction $m \geq r$ and $n \leq s$, the summation on the right-hand side of (1.6) reduces to

$$H^{\beta,\gamma}(m, n|r, s) - H^{\beta,\gamma}(m-1, n|r-1, s) - H^{\beta,\gamma}(m, n-1|r, s-1) = \binom{\beta+m+n}{\beta+m} \binom{\gamma+r+s}{\gamma+s}, \quad m \geq r \text{ and } n \leq s. \quad (1.7)$$

We may also deduce another result of the form

$$H^{\alpha,\beta}(m, n|r, s) - H^{\alpha,\beta}(m-1, n|r-1, s) - H^{\alpha,\beta}(m, n-1|r, s-1) = \binom{r+s}{s} \sum_{k=0}^m \binom{\alpha+k-1}{k} \binom{\beta+m+n-k}{\beta+m-k} \quad (1.8)$$

provided $m \leq r$ or $n \leq s$, where

$$H^{\alpha,\beta} = \sum_{j=0}^{\min(n,s)} \sum_{i=0}^{\min(m,r)} \binom{i+j}{i} \binom{\alpha+m-i+j}{m-i} \cdot \binom{\beta+n-j+i}{n-j} \binom{r+s-i-j}{r-i}. \quad (1.9)$$

Evidently $\alpha = \beta = 0$ in (1.8) and $\gamma = \beta = 0$ in (1.6) will give the Carlitz expression in (1.2).

Now recently Carlitz [9] reconsidered his identity (1.2) and studied it for

the case $m > r$ and $n > s$. In the process he derived, as a main result, the expression

$$H(m, n|r, s) - H(m-1, n|r-1, s) - H(m, n-1|r, s-1) \\ = \binom{r+s}{r} \sum_{i=0}^s \binom{i+m-r-1}{i} \binom{n+r-i}{r}. \quad (1.10)$$

A special case of our theorem gives

$$H^{\beta, \gamma}(m, n|r, s) - H^{\beta, \gamma}(m-1, n|r-1, s) - H^{\beta, \gamma}(m, n-1|r, s-1) \\ = \binom{\gamma+r+s}{\gamma+s} \sum_{i=0}^s \binom{i+m-r-1}{i} \binom{\beta+r+n-i}{\beta+r} \quad (1.11)$$

for $m > r$ and $n > s$. Evidently $\beta = \gamma = 0$ in (1.11) results in the formula (1.10).

The expressions (1.6) and (1.11) permit us to make the following statement:

$$H^{\beta, \gamma}(m, n|r, s) - H^{\beta, \gamma}(m-1, n|r-1, s) - H^{\beta, \gamma}(m, n-1|r, s-1) > 0 \quad (1.12)$$

for $\beta, \gamma > 0$ and m, n, r, s nonnegative integers, where $H^{\beta, \gamma}$ is defined by (1.5). The special case $\beta = \gamma = 0$ was first given as a theorem by Carlitz [9, equation (18)].

For β a nonnegative integer, γ real or complex, we deduce a new expression

$$H^{\beta, \gamma}(2r + \beta + 1, 2s + 1|r, s) - H^{\beta, \gamma}(2r + \beta, 2s + 1|r-1, s) \\ - H^{\beta, \gamma}(2r + \beta + 1, 2s|r, s-1) \\ = \frac{1}{2} \binom{2\beta + 2r + 2s + 2}{2\beta + 2r + 1} \binom{\gamma + r + s}{\gamma + s}. \quad (1.13)$$

Again, for $\beta = \gamma = 0$, the known Carlitz result [9, equation (21)] presents itself. Note the misprints in Carlitz [9, equations (20), (21)].

The above special cases are some one-dimensional examples of the theorem. As an illustration, we give one two-dimensional special case of our multi-dimensional expression. The two-dimensional analog of (1.12) is

$$F(m, n|r, s) - F(m-1, n|r-1, s) - F(m, n-1|r, s-1) > 0 \quad (1.14)$$

for $2b-1 > 0$, $2c-1 > 0$, and m, n, r, s nonnegative integers, where

$$F(m, n, r, s) = \sum_{s_1=0}^s \sum_{r_1=0}^r \sum_{m_1=0}^m \sum_{n_1=0}^n f(m_1, n_1|r_1, s_1) \\ \cdot f(m-m_1, n-n_1|r-r_1, s-s_1) \quad (1.15)$$

and

$$\begin{aligned}
 f(m, n|r, s) = & \sum_{j=0}^{\min(n,s)} \sum_{i=0}^{\min(m,r)} \binom{c+i+j-1}{i} \binom{j-\frac{1}{2}}{j} \\
 & \cdot \binom{m-i+j-\frac{1}{2}}{m-i} \binom{b+n-j+i-1}{n-j} \\
 & \cdot \binom{r+s-i-j-\frac{1}{2}}{r-i} \binom{s-j-\frac{1}{2}}{s-j}. \quad (1.16)
 \end{aligned}$$

The multi-dimensional theorem is presented and derived in the next section.

II. THEOREM. For a, b, c real or complex numbers, m, n, r, s nonnegative integers, and σ a positive integer

$$\begin{aligned}
 F(m, n|r, s) - F(m-1, n|r-1, s) - F(m, n-1|r, s-1) \\
 = G(m, n|r, s) - H(m, n|r, s), \quad (2.1)
 \end{aligned}$$

where

$$\begin{aligned}
 G = & \sum_{k=0}^{\min(m,r)} \sum_{p=0}^n \binom{a\sigma + n + m - p - k - 1}{m-k} \binom{r+s-k}{s} \\
 & \cdot \binom{c\sigma + k - 2}{k} \binom{b\sigma + k + p - 2}{p}, \quad (2.2)
 \end{aligned}$$

$$\begin{aligned}
 H = & \binom{c\sigma - 1 + r + s}{c\sigma - 1 + s} \sum_{l=0}^{n-s-1} \binom{b\sigma - 1 + r + l}{l} \binom{a\sigma - 2 + m + n - r - l}{m-r-1}, \quad (2.3)
 \end{aligned}$$

where $H = 0$ for $m \leq r$ or $n \leq s$,

$$\begin{aligned}
 F(m, n|r, s) = & \sum_{\substack{n_1, n_2, \dots, n_\sigma \\ n_i \geq 0 \\ n_1 + n_2 + \dots + n_\sigma = n}} \sum_{\substack{m_1, \dots, m_\sigma \\ m_i \geq 0 \\ m_1 + \dots + m_\sigma = m}} \sum_{\substack{r_1, \dots, r_\sigma \\ r_i \geq 0 \\ r_1 + \dots + r_\sigma = r}} \\
 & \cdot \sum_{\substack{s_1, \dots, s_\sigma \\ s_i \geq 0 \\ s_1 + \dots + s_\sigma = s}} \prod_{i=1}^{\sigma} f(m_i, n_i|r_i, s_i), \quad (2.4)
 \end{aligned}$$

$$\begin{aligned}
 f(m, n|r, s) = & \sum_{j=0}^{\min(n,s)} \sum_{i=0}^{\min(m,r)} \binom{c+i+j-1}{i} \binom{1/\sigma + j - 1}{j} \\
 & \cdot \binom{a+m-i+j-1}{m-i} \binom{b+n-j+i-1}{n-j} \\
 & \cdot \binom{1/\sigma + r+s-i-j-1}{r-i} \binom{1/\sigma + s-j-1}{s-j} \quad (2.5)
 \end{aligned}$$

and

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}, \quad \text{quotient of gamma functions.}$$

COROLLARY 1. $F_1(m, n|r, s)$ is obtained by letting $a = 1/\sigma$ in (2.4) and (2.5).
Then

$$\begin{aligned} F_1(m, n|r, s) - F_1(m-1, n|r-1, s) - F_1(m, n-1|r, s-1) \\ = G_1(m, n|r, s) - H_1(m, n|r, s) \end{aligned} \quad (2.6)$$

where

$$G_1 = \binom{b\sigma - 1 + m + n}{n} \sum_{k=0}^{\min(m,r)} \binom{r+s-k}{s} \binom{c\sigma + k - 2}{k} \quad (2.7)$$

and H_1 is obtained from (2.3) by letting $a = 1/\sigma$.

COROLLARY 2. With the restrictions $m > r$ and $n > s$ in Corollary 1

$$\begin{aligned} F_1(m, n|r, s) - F_1(m-1, n|r-1, s) - F_1(m, n-1|r, s-1) \\ = G_2(m, n|r, s) \end{aligned} \quad (2.8)$$

where

$$G_2 = \binom{c\sigma - 1 + r + s}{c\sigma - 1 + s} \sum_{l=0}^s \binom{b\sigma - 1 + r + n - l}{b\sigma - 1 + r} \binom{m - r + l - 1}{l} \quad (2.9)$$

which is the combination of G and H .

COROLLARY 3. $F_3(m, n|r, s)$ is obtained by letting $c = 1/\sigma$ in (2.4) and (2.5).
Then

$$\begin{aligned} F_3(m, n|r, s) - F_3(m-1, n|r-1, s) - F_3(m, n-1|r, s-1) \\ = G_3(m, n|r, s) - H_3(m, n|r, s) \end{aligned} \quad (2.10)$$

where

$$G_3 = \binom{r+s}{s} \sum_{p=0}^n \binom{a+m+p-1}{a+p-1} \binom{b+n-p-2}{n-p} \quad (2.11)$$

and H_3 is defined in (2.3) with $c = 1/\sigma$.

COROLLARY 4. For $b\sigma - 1 > 0$, $c\sigma - 1 > 0$, m, n, r, s nonnegative integers

$$F_1(m, n|r, s) - F_1(m-1, n|r-1, s) - F_1(m, n-1|r, s-1) > 0 \quad (2.12)$$

where F_1 is defined in Corollary 1.

COROLLARY 5. For $b\sigma - 1$ a nonnegative integer, c real or complex

$$\begin{aligned} F_1(2r + b\sigma, 2s + 1|r, s) - F_1(2r + b\sigma - 1, 2s + 1|r - 1, s) \\ - F_1(2r + b\sigma, 2s|r, s - 1) \\ = \frac{1}{2} \binom{2b\sigma + 2r + 2s}{2b\sigma + 2r - 1} \binom{c\sigma + r + s - 1}{c\sigma + s - 1} \end{aligned} \quad (2.13)$$

where F_1 is defined in Corollary 1.

PROOF OF THE THEOREM. NOW

$$\sum_{m,n,r,s=0}^{\infty} x^m y^n u^r v^s f(m, n|r, s) \quad (2.14)$$

$$= \sum_{m,n,r,s,i,j=0}^{\infty} \frac{x^m y^n u^r v^s (ux)^i (vy)^j (c+j)_i (1/\sigma)_j (a+j)_m (b+i)_n (1/\sigma)_{r+s}}{j! i! m! n! r! s!} \quad (2.15)$$

$$= \frac{(1-x)^{1/\sigma-a} (1-y)^{c-b} (1-ux-y)^{1/\sigma-c}}{(1-u-v)^{1/\sigma} [(1-x)(1-y) - ux(1-x) - vy(1-y)]^{1/\sigma}}. \quad (2.16)$$

(2.15) is derived from (2.14) by multiple series transformations, and (2.16) is subsequently deduced by simplifying the individual series.

(2.16) may be expressed as

$$\begin{aligned} (1-x)^{1/\sigma-a} (1-y)^{c-b} (1-ux-y)^{1/\sigma-c} \\ \cdot \left\{ \frac{1}{(1-ux-vy)(1-x-y)} \right. \\ \left. \cdot \left[\frac{1}{1-u-v} - \frac{xy}{(1-x)(1-y) - ux(1-x) - vy(1-y)} \right] \right\}^{1/\sigma}. \end{aligned} \quad (2.17)$$

Equating (2.14) with (2.17), and raising both sides to the power σ , one has

$$\begin{aligned} (1-x)^{1-a\sigma} (1-y)^{c\sigma-b\sigma} (1-ux-y)^{1-c\sigma} \\ \cdot \left\{ \frac{1}{1-x-y} \right. \\ \left. \cdot \left[\frac{1}{1-u-v} - \frac{xy}{(1-x)(1-y) - ux(1-x) - vy(1-y)} \right] \right\} \end{aligned} \quad (2.18)$$

$$= (1-ux-vy) \left\{ \sum_{m,n,r,s=0}^{\infty} x^m y^n u^r v^s f(m, n|r, s) \right\}^{\sigma}. \quad (2.19)$$

(2.18) is treated in two parts. Consider

$$\frac{(1-x)^{1-a\sigma}(1-y)^{1-b\sigma}\{1-ux/(1-y)\}^{1-c\sigma}}{(1-x-y)(1-u-v)} \quad (2.20)$$

$$= \sum_{i,j,k,m_1,m_2,n_1,n_2=0}^{\infty} \frac{x^{m_1+k+i}y^{m_2+j}u^{n_1+m}v^{n_2}(c\sigma-1)_{m_1}(i+j)!(n_1+n_2)!}{m_1!i!j!n_1!n_2!} \\ \cdot \frac{(a\sigma-1)_k(m_1+b\sigma-1)_{m_2}}{k!m_2!} \quad (2.21)$$

$$= \sum_{m,n,r,s=0}^{\infty} x^m y^n u^r v^s \sum_{m_1,m_2,k=0}^{\infty} \frac{(c\sigma-1)_{m_1}(m+n-k-m_1-m_2)!}{m_1!(m-k-m_1)!(n-m_2)!} \\ \cdot \frac{(r+s-m_1)!(a\sigma-1)_k(m_1+b\sigma-1)_{m_2}}{(r-m_1)!s!k!m_2!} \quad (2.22)$$

$$= \sum_{m,n,r,s=0}^{\infty} x^m y^n u^r v^s \sum_{p=0}^n \sum_{k=0}^{\min(m,r)} \binom{a\sigma+n+m-p-k-1}{m-k} \\ \cdot \binom{r+s-k}{s} \binom{c\sigma+k-2}{k} \binom{b\sigma+k+p-2}{p}. \quad (2.23)$$

(2.21) is the expansion of (2.20), (2.22) involves series transformations, and (2.23) is deduced by reducing one of the series. This completes the expansion of the first part of (2.18).

The second part of (2.18)

$$\frac{(1-x)^{1-a\sigma}(1-y)^{c\sigma-b\sigma}(1-ux-y)^{1-c\sigma}}{1-x-y} \\ \cdot \left\{ \frac{xy}{(1-x)(1-y)-ux(1-x)-vy(1-y)} \right\} \quad (2.24)$$

may be expressed as

$$xy \sum_{n_1,n_2,i,j,p=0}^{\infty} \frac{(n_1+n_2)!(i+j)!(c\sigma-1)_p(1-x)^{-a\sigma-j}(1-y)^{-b\sigma-i-p}}{n_1!n_2!i!j!p!} \\ \cdot x^{n_1+i+p}y^{n_2+j}u^{i+p}v^j \quad (2.25)$$

$$= xy \sum_{m,n,r,s=0}^{\infty} x^m y^n u^r v^s \sum_{l=0}^{n-s-1} \\ \cdot \frac{(b\sigma+r)_l(-1)^m(1-c\sigma-r-s)_r(-a\sigma-m-n+r+l)_{m-r}}{l!(m-r)!r!} \quad (2.26)$$

(2.18) is now equal to the difference between (2.23) and (2.26). Equating (2.19) with this difference, and comparing coefficients, results in the identity (2.1).

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