PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 2, September 1979

RATIONAL SURFACES WITH TOO MANY VECTOR FIELDS

GERALD NEUMAN

ABSTRACT. A method is given for constructing smooth rational surfaces with nonreduced automorphism groups, by a sequence of blow-ups of the projective plane. The technique works in all positive characteristics.

1. Introduction. The vector space $V_X = H^0(X, \Theta_X)$ of regular vector fields on a surface X can be identified with the tangent space to the group scheme Aut X at its identity element. Thus a surface X has a nonreduced automorphism group scheme iff dim $V_X >$ dim Aut X. We wish to describe a method for constructing smooth rational surfaces with this property. Since group schemes in characteristic zero are smooth [**1**, p. 101], this will be possible only for positive characteristics. (In the language of deformation theory, V_X may also be identified with the set of infinitesimal automorphisms of X parameterized by the scheme Spec $k[t]/(t^2)$. If dim $V_X >$ dim Aut X, then X has "obstructed" infinitesimal automorphisms, that is, infinitesimal automorphisms which do not extend to an algebraic family of automorphisms.)

If X is a smooth surface, and Y is the blow-up of X at a smooth point P, then the relationship between Aut X and Aut Y can be summarized in two rules.

LEMMA (1.1). *The identity component* $\mathrm{Aut}^0\, Y$ *is a closed subgroup of* Aut X, *whose support is the identity component of the subgroup of automorphisms which fix* P.

LEMMA (1.2). *The tangent space* V_Y *is the subspace of* V_X *consisting of the vector fields which vanish at* P.

Lemma (1.2) is the first step in the proof of Lemma (1.1). It follows from Lemma (3.1) below. If we let C denote the closed subscheme of Aut X representing the functor C: $T \mapsto \{T$-Automorphisms of $X \times T$ fixing the closed subscheme $P \times T\}$, then it is not difficult to show that $\mathrm{Aut}^0\, Y$ is isomorphic to the identity component C^0. Details may be found in [3].

Thus our strategy is to blow up at a point which is fixed by too few automorphisms, or, equivalently, at which too many vector fields vanish.

The projective plane has a reduced automorphism group, namely PGL(2). We will make a series of blowings-up of the plane, each time at a point

Received by the editors January 9, 1978.
AMS (*MOS*) *subject classifications* (1970). Primary 14J10, 14L15.
Key words and phrases. Rational surface, automorphism group scheme, regular vector field.

infinitely near the previous point, that is, on the new exceptional divisor. It is convenient to use multi-projective coordinates to describe such a procedure.

Let B_0 denote the projective plane $\mathbf{P}^2$ with coordinates (X_0, X_1, X_2).

Let P_0 be the point $(1, 0, 0)$. We can also take inhomogeneous coordinates $x_1 = X_1/X_0$ and $x_2 = X_2/X_0$ at P_0. The blow-up B_1 of the surface B_0 at the point P_0 can be viewed as the subvariety of $B_0 \times \mathbf{P}^1$ consisting of all points $(X_0, X_1, X_2; Y_0, Y_1)$ for which $X_1Y_1 = X_2Y_0$. In inhomogeneous coordinates, with $y_1 = Y_1/Y_0$, this gives the familiar local equation $x_1y_1 = x_2$.

Choose a point $P_1 = (1, 0, 0; 1, \lambda_1)$ on the exceptional divisor. The blow-up of B_1 at this point consists of all points of $B_1 \times \mathbf{P}^1$ of the form $(X_0, X_1, X_2; Y_0, Y_1; Z_0, Z_1)$ satisfying the equation

$$X_1Y_0Z_1 = (Y_1 - \lambda_1Y_0)Z_0X_0.$$

Once more, taking the inhomogeneous coordinate $y_2 = Z_1/Z_0$ gives the local equation $x_1y_2 = y_1 - \lambda_1$.

Similarly, for any choice of $\lambda_1, \ldots, \lambda_k$, we can take a point $P_k = (1, 0, 0; 1, \lambda_1; \cdots; 1, \lambda_k)$ on B_k and blow up to get a surface B_{k+1} defined in $B_k \times \mathbf{P}^1$ as the set of all $(X_0, X_1, X_2; Y_0, Y_1; \cdots; V_0, V_1; W_0, W_1)$ for which

$$X_1V_0W_1 = (V_1 - \lambda_kV_0)W_0X_0,$$

and with $y_{k+1} = W_1/W_0$ and $y_k = V_1/V_0$, this gives the local equation $x_1y_{k+1} = y_k - \lambda_k$. With this notation, we will investigate how Aut B_{k+1} and dim $H^0(B_{k+1}, \Theta_{B_{k+1}})$ depend on the choice of $\lambda_1, \ldots, \lambda_k$. We can take $\lambda_1 = 0$ without loss of generality.

2. Calculation of automorphisms. We wish to compute the dimension of Aut B_{k+1}. By Lemma (1.1), it suffices to determine which automorphisms of B_k fix the point P_k. Applying the lemma repeatedly, we can view an element α of the identity component $\mathrm{Aut}^0\, B_k$ as an automorphism of $\mathbf{P}^2$. If α fixes P_k, it must send each curve passing through P_k to another curve with the same property. This condition can be stated on $\mathbf{P}^2$: we ask that α send each curve whose proper transform on B_k passes through P_k to another such curve.

Given $\lambda_1, \ldots, \lambda_k$, we put

$$F_k(T) = \lambda_2T^2 + \cdots + \lambda_kT^k.$$

If S_k denotes Spec $k[t]/(t^{k+1})$, we define

$$f_k\colon S_k \to \mathbf{P}^2$$

by

$$t \leftarrow x_1, \qquad F_k(t) \leftarrow x_2.$$

With this notation, a straightforward calculation proves the following lemma.

Lemma (2.1). *Let C be a smooth curve on $\mathbf{P}^2$. Then the proper transform of C on B_k passes through P_k iff f_k factors through C.*

Thus to fix P_k, an automorphism of $\mathbf{P}^2$ must commute with f_k up to an automorphism of S_k.

We recall that every automorphism α of $\mathbf{P}^2$ can be written as a linear map

$$\alpha\colon \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} aX_0 & + & bX_1 & + & cX_2 \\ dX_0 & + & eX_1 & + & fX_2 \\ gX_0 & + & hX_1 & + & iX_2 \end{bmatrix}$$

and that two linear maps give the same automorphism if they differ by a scalar. We wish to determine the conditions imposed on the coefficients of the matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

when we require α to fix P_k.

To fix P_0 at all, we must have $d = g = 0$. If α also fixes P_1, it must fix the line $X_2 = 0$, and so h is likewise zero. Then the matrix is invertible and upper triangular, and its diagonal entries a, e, and i are all nonzero. We can now eliminate the ambiguity about scalars by choosing $a = 1$.

Then α commutes with f_k up to an automorphism of S_k iff

$$F_k\left(\frac{eu + fF_k(u)}{1 + bu + cF_k(u)}\right) \equiv \frac{iF_k(u)}{1 + bu + cF_k(u)} \quad \bmod u^{k+1}. \tag{2.2}$$

An example. Rather than describe the conditions imposed on the entries b, c, e, f, and i by equation (2.2) in general, we would like to work out a specific example, which gives us in each positive characteristic one of our rational surfaces with nonreduced automorphism group.

Let the characteristic of the ground field be p. Then we will take $\lambda_1 = \lambda_2 = \cdots = \lambda_{2p} = 0$, $\lambda_{2p+1} = \lambda \neq 0$, $\lambda_{2p+2} = \cdots = \lambda_{3p} = 0$, and $\lambda_{3p+1} = \mu \neq 0$.

Then the condition that α fix P_{3p+1} is

$$\begin{aligned} &\lambda\left(\frac{eu + f\lambda u^{2p+1} + f\mu u^{3p+1}}{1 + bu + c\lambda u^{2p+1} + c\mu u^{3p+1}}\right)^{2p+1} \\ &\qquad + \mu\left(\frac{eu + f\lambda u^{2p+1} + f\mu u^{3p+1}}{1 + bu + c\lambda u^{2p+1} + c\mu u^{3p+1}}\right)^{3p+1} \\ &\equiv \frac{i\lambda u^{2p+1} + i\mu u^{3p+1}}{1 + bu + c\lambda u^{2p+1} + c\mu u^{3p+1}} \quad \bmod u^{3p+2}. \end{aligned}$$

The congruence breaks into the two equations:

$$i\lambda = \lambda e^{2p+1}$$

and

$$3b^p i\lambda + u\mu = \mu e^{3p+1} + b^p e^{2p+1}\lambda;$$

solving simultaneously, the latter yields

$$\mu e^{2p+1}(1 - e^p) = \lambda e^{2p+1} b^p(1 - 3).$$

Clearly these give two conditions on the coefficients, and the automorphism group thus has dimension three; we will later see that this choice of λ's imposes only one condition on the tangent fields, giving a tangent space of dimension four.

3. Calculation of vector fields. Given our sequence of surfaces $B_1, \ldots, B_{k+1}$, we wish to calculate the dimension of $V_{B_{k+1}} = H^0(B_{k+1}, \Theta_{B_{k+1}})$ by determining inductively which vector fields on B_j vanish at P_j. To do so, we will first take a vector field vanishing at P_{j-1} and find an expression for the vector field to which it lifts on B_j.

We can take x_1 and $y_{j-1} - \lambda_{j-1}$ as parameters at the point P_{j-1}. Any vector field θ_{j-1} on B_{j-1} can then be written in the form $fd/dx_1 + gd/dy_{j-1}$. If it lifts to a vector field θ_j on B_j, we wish to find an expression for θ_j in the form $Fd/dx_1 + Gd/dy_j$.

LEMMA (3.1). *The vector field θ_{j-1} lifts to a regular vector field θ_j on B_j iff it vanishes at P_{j-1}, and then at P_j we can write*

$$\theta_j = f\frac{d}{dx_1} + g_j\frac{d}{dy_j},$$

with $g_j = (1/x_1)(g - fy_j)$.

PROOF. Locally at P_{j-1}, the surface B_{j-1} is an open subset of Spec $k[x_1, y_{j-1}]$, and the blowing-up corresponds to the ring homomorphism

$$k[x_1, y_{j-1}] \xrightarrow{\phi} k[x_1, y_{j-1}, y_j]/(y_{j-1} - \lambda_{j-1} - x_1 y_j).$$

We may view the vector fields as derivations on these rings, and we will let D_w denote d/dw on $k[x_1, y_{j-1}]$, and Δ_w denote d/dw on $k[x_1, y_{j-1}, y_j]/(y_{j-1} - \lambda_{j-1} - x_1 y_j)$.

Then

$$D_{x_1} = \Delta_{x_1} \circ \phi - \frac{y_j}{x_1}\Delta_{y_j} \circ \phi \quad \text{and} \quad D_{y_j} = \frac{1}{x_1}\Delta_{y_j} \circ \phi.$$

We conclude that $fd/dx_1 + gd/dy_{j-1}$ must lift to the vector field

$$f\frac{d}{dx_1} + \frac{1}{x_1}(g - fy_j)\frac{d}{dy_j}.$$

A priori, a vector field of this form has coefficients in the function field $k(x_1, y_j)$; for θ_j to be regular on our open subset of B_j, x_1 must divide $g - fy_j$. This occurs iff g has no constant term, that is, iff g vanishes at P_{j-1}. By symmetry, θ_j is regular on all of B_j iff both f and g vanish at P_{j-1}. That is, θ_{j-1} must vanish at P_{j-1}. This proves the lemma.

(Since the question is local in the étale topology, this also proves Lemma (1.2).)

We suppose now that θ_{j-1} vanishes at P_{j-1}, and so lifts to θ_j on B_j. When does θ_j vanish at P_j? The function f already vanishes at P_{j-1}, and therefore vanishes on the entire exceptional divisor over P_{j-1}. It remains to examine $g_j = (1/x_1)(g - fy_j)$. If we break $g - fy_j$ into a sum of homogeneous parts with respect to x_1,

$$g - fy_j = H_0 + H_1 x_1 + H_2 x_1^2 + \dots,$$

we see that H_0 must vanish at P_j, and that g_j vanishes at P_j iff H_1 does.

Thus the vector field θ_j vanishes at P_j iff the coefficient H_1 of the part of x_1-degree one in $g - fy_j$ does.

Returning to B_0, we recall that $H^0(\mathbf{P}^2, \Theta_{\mathbf{P}^2})$ is spanned by the fields $X_i d/dX_j$, with the single relation

$$X_0 \frac{d}{dX_0} + X_1 \frac{d}{dX_1} + X_2 \frac{d}{dX_2} = 0.$$

This relation permits us to express $X_i d/dX_0$ as

$$X_i \frac{d}{dX_0} = \frac{X_i}{X_0} X_0 \frac{d}{dX_0} = -\frac{X_i X_1}{X_0}\frac{d}{dX_1} - \frac{X_i X_2}{X_0}\frac{d}{dX_2}.$$

Since

$$X_0 \frac{d}{dX_j} = \frac{d}{d(X_j/X_0)},$$

we can write a basis in parameters at (1, 0, 0) in the form

$$\begin{aligned}
\alpha &= X_0 \frac{d}{dX_0} = -x_1 \frac{d}{dx_1} - x_2 \frac{d}{dx_2},\\
\beta &= X_1 \frac{d}{dX_0} = -x_1^2 \frac{d}{dx_1} - x_1 x_2 \frac{d}{dx_2},\\
\gamma &= X_2 \frac{d}{dX_0} = -x_1 x_2 \frac{d}{dx_1} - x_2^2 \frac{d}{dx_2},\\
\delta &= X_0 \frac{d}{dX_1} = \frac{d}{dx_1},\\
\varepsilon &= X_1 \frac{d}{dX_1} = x_1 \frac{d}{dx_1},\\
\zeta &= X_2 \frac{d}{dX_1} = x_2 \frac{d}{dx_1},\\
\eta &= X_0 \frac{d}{dX_2} = \frac{d}{dx_2},\\
\theta &= X_1 \frac{d}{dX_2} = x_1 \frac{d}{dx_2}.
\end{aligned}$$

The fields α, β, γ, ε, ζ, and θ vanish at (1, 0, 0) and lift to B_1; all of these but θ vanish at P_1, and lift to B_2.

Lemma (3.1) allows us to compute the form a vector field on B_{j-1} takes on B_j; this recursive procedure also yields an explicit equation. If a vector field has the form $fd/dx_1 + g_k d/dy_k$ at P_k for each k, then the g_k are related by the formula $g_k = (1/x_1)(g_{k-1} - fy_k)$, and so

$$g_k = \frac{g_1}{x_1^{k-1}} - f\sum_{r=2}^{k} \frac{y_1}{x_1^{k-r+1}}. \tag{3.2}$$

Using the defining equation for y_j, $y_j x_1 = y_{j-1} - \lambda_{j-1}$, we can compute that

$$\frac{y_j}{x_1^n} = y_{j+n} + \sum_{m=0}^{n-1} \frac{\lambda_{m+j}}{x_1^{n-m}}. \tag{3.3}$$

It then follows that

$$\sum_{r=2}^{k} \frac{y_1}{x_1^{k-r+1}} = (k-1)\frac{y_k}{x_1} + \sum_{m=0}^{k-1} (m-1)\frac{\lambda_m}{x_1^{k-m+1}}. \tag{3.4}$$

Since we are interested in $H_1 x_1$, the term of x_1-degree one in $g_{k-1} - fy_k$, we combine formulae (3.2) and (3.4) in the single equation

$$\begin{aligned} g_{k-1} - fy_k &= x_1 g_k \\ &= \frac{g_1}{x_1^{k-2}} - (k-1)fy_k - f\sum_{m=1}^{k-1} (m-1)\frac{\lambda_m}{x_1^{k-m}}. \end{aligned} \tag{3.5}$$

Then by inspection, using (3.3) in the form

$$y_1 = x_1^{k-1}\left(y_k + \sum_{m=0}^{k-2} \frac{\lambda_{m+1}}{x_1^{k-m-1}} \right),$$

we can easily fill in the following table.

TABLE (3.6)

Field	f	g_1	$H_1 x_1$ (for $k > 1$)
α	$-x_1$	0	$(k-1)y_k x_1$
β	$-x_1^2$	0	$(k-2)\lambda_{k-1}x_1$
γ	$-x_1^2 y_1$	0	$\Sigma_{m=2}^{k-1}(m-1)\lambda_m\lambda_{k-m}x_1$
ε	x_1	$-y_1$	$-ky_k x_1$
ζ	$x_1 y_1$	$-y_1^2$	$\Sigma_{m=2}^{k-1} m\lambda_m\lambda_{k-m+1}x_1$

As we have seen, a vector field will vanish at P_k iff the associated H_1 does. We note that the H_1 terms for β, γ, and ζ do not depend on the choice of λ_k, and so if these fields vanish for some P_k they must vanish on the entire exceptional divisor. The field β vanishes on the divisor if either $\lambda_{k-1} = 0$, or $k \equiv 2 \bmod p$. On the other hand, α and ε only vanish for $\lambda_k = 0$, or $k \equiv 1 \bmod p$ (for α), or $k \equiv 0 \bmod p$ (for ε). In these cases, α and ε vanish on the entire exceptional divisor.

Suppose $\lambda_1, \ldots, \lambda_n = 0$. Then the fields γ and ζ will vanish on the exceptional divisor for all $k < 2n + 1$.

Applying these observations to the example discussed in §2 above, we recall that the only nonzero λ_k occurred for $k = 2p + 1$ and $k = 3p + 1$. Thus, α, β, γ, and ζ all vanish at P_k for all $k < 3p + 1$, and $V_{B_{3p+2}}$ has dimension four. But as we saw, Aut B_{3p+2} has dimension three. This provides the desired example.

4. Alternatives. The methods used to calculate automorphisms and vector fields, although applied here to a specific example, are general enough to describe many others. The simplest variation would be to postpone the second nonzero λ to some distant step, also congruent to 1 mod p; the same result will follow. Instead of using the fields α and β, we could take the only nonzero λ's at steps congruent to 0 mod p, and construct an example using the field ε.

Finally, we can extend the present example until there are no automorphisms left, and still have vector fields. For example, in characteristic 2, if the only nonzero λ's are at $k = 5, 7, 9, 13$, and 17, then equation (2.2) shows that the only automorphism fixing P_{17} is the identity ($e = i = 1$, $b = c = f = 0$), but the vector fields α and β both lift to B_{17} and vanish at P_{17}. (Of course, characteristic 2 is a little special here, because the expressions $(k - 1)\lambda_k$ and $(k - 2)\lambda_{k-1}$ can only be synchronized in characteristic 2. Normally, we can only arrange for a single vector field (e.g., α) to lift to B_n with discrete automorphism group.)

References

1. D. Mumford, *Abelian varieties*, Oxford University Press, Bombay, 1970.

2. A. N. Rudakov and I. R. Shafarevich, *Inseparable morphisms of algebraic surfaces*, Izv. Akad. Nauk SSSR Ser. Math. **40** (1976), 1269–1307. (Russian)

3. G. Neuman, *Extra vector fields on algebraic surfaces*, Thesis, Massachusetts Institute of Technology, Cambridge, Mass., 1977.

Current address: 3 Langdon Street, Apartment 3, Cambridge, Massachusetts 02138