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THE SCHUR INDEX OF THE p-REGULAR CHARACTERS OF THE BOREL SUBGROUP

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ABSTRACT. Let B be the group of $\mathbf{F}_q$-rational points of the Borel subgroup of a connected reductive group defined over the finite field $\mathbf{F}_q$. It is shown that under appropriate conditions, all irreducible characters of B which have degree prime to q have Schur index one.

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over the finite field $\mathbf{F}_q$, with corresponding Frobenius map F. We assume that $\mathbf{G}$ has connected centre $\mathbf{Z}$ and that the characteristic p is good for $\mathbf{G}$. If $\mathbf{H}$ is a closed subgroup of $\mathbf{G}$, H will denote its group of $\mathbf{F}_q$-rational points; for a finite group H, a complex irreducible character is p-regular if p does not divide its degree; $m_p(H)$ is the number of such characters. It was shown in [2] that if $\mathbf{B}$ is an F-stable Borel subgroup of $\mathbf{G}$, then $m_p(B) = m_p(G)$, verifying a special case of Alperin's conjecture for finite groups. Ohmori [3] has shown that the $|Z|q^l$ (where l is the semisimple rank of $\mathbf{G}$) p-regular characters of G all have Schur index one, i.e. can be realized in their field of characters. In this note we prove the corresponding result for p-regular characters of B, viz:

THEOREM. *The p-regular characters of B all have Schur index one.*

In the proof, we use the notation of [1] and [2]. $\mathbf{B}$ is the semidirect product $\mathbf{T} \ltimes \mathbf{U}$, where $\mathbf{U}$ is a maximal unipotent subgroup of $\mathbf{G}$. Correspondingly, $B = T \ltimes U$, where U is a p-group and T an abelian p'-group.

PROPOSITION 1. *Let χ be a p-regular character of B. Then $\chi = (\mu\phi)^B$, where μ is a linear character of U and ϕ is a character of the centralizer $T(\mu)$ of μ in T. (Here $\mu\phi$ is a character of $T(\mu) \ltimes U$).*

This is elementary and was noted in [2].

COROLLARY 1′. (i) $\chi|_{T(\mu)\cdot U} = \varphi \cdot \sum_{t \in T/T(\mu)} \mu^t$;
(ii) *χ vanishes outside $T(\mu) \cdot U$.*

PROOF. (i) is a simple application of Frobenius' formula for induced characters, and (ii) follows since $T(\mu) \cdot U$ is normal in B.

PROPOSITION 2. *$\sum_{t \in T/T(\mu)} \mu^t$ takes rational values on U.*

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PROOF. We have a canonical isomorphism [1, p. 258] η: $U/U' \to X_1 \times \cdots \times X_s$, where $X_i \cong (\mathbf{F}_{q^{n_i}})^+$. Thus we speak of the *support* of μ, defined as $\{i \mid \mu(X_i) \neq 1\}$. Because of our assumption of connected centre, T acts transitively on the set of linear characters of U with given support (by the argument used to prove Theorem B′ in [1]). Hence $\Sigma_{t \in T/T(\mu)} \mu^t$ is the sum of *all* linear characters of U with fixed support I. By applying Galois automorphisms, one sees that this sum always takes rational values.

For any character ξ of some finite group, denote by $\mathbf{Q}(\xi)$ the algebraic extension of $\mathbf{Q}$ obtained by adjoining all the values of ξ. This is the character field of ξ. From Proposition 2 we have immediately:

COROLLARY 2′. *If χ is a p-regular character of B, and ϕ is as in Proposition* 1, *then* $\mathbf{Q}(\chi) = \mathbf{Q}(\phi)$.

LEMMA 3. *The restriction of χ to T is* $\phi^T_{T(\mu)}$.

PROOF. This follows directly from Corollary 1′ by evaluation of $\chi|_T$, or by applying Mackey's subgroup theorem.

LEMMA 4. $\phi_{T(\mu)}$ *has an extension $\bar{\phi}$ to T, such that* $\mathbf{Q}(\bar{\phi}) = \mathbf{Q}(\phi)$.

PROOF. If Z is trivial then $T = T_1 \times T(\mu)$, since T is a direct product of the groups $\mathbf{F}^*_{q^{n_i}}$ and the condition $t \in T(\mu)$ is that certain components be trivial. In the general case, since the $\mathbf{Z}$-span of the fundamental roots has a Frobenius-invariant complement in the character group $X(\mathbf{T})$, we have $\mathbf{T} \cong \mathbf{Z} \times \mathbf{T}/\mathbf{Z}$; using Lang's theorem it follows that $T \cong Z \times T/Z$. Hence $T(\mu)$ is again a direct factor (containing Z) and if $T = T(\mu) \times T_1$, we may take $\bar{\varphi}(t, t_1) = \varphi(t)$.

COROLLARY 4′. *The multiplicity* $(\chi, \bar{\phi}^B_T) = 1$.

PROOF. From Lemmas 3 and 4, $(\chi, \bar{\varphi})_T = (\varphi^T_{T(\mu)}, \bar{\varphi}) = 1$. The corollary now follows by Frobenius reciprocity.

PROOF OF THE THEOREM. Let χ, φ be as in Proposition 1. From Corollary 2′ we have $\mathbf{Q}(\chi) = \mathbf{Q}(\varphi)$. The theorem is therefore proved if χ can be realized over $\mathbf{Q}(\varphi)$. But (with $\bar{\varphi}$ as in Lemma 4) $\bar{\varphi}^B_T$ is a representation of B in $\mathbf{Q}(\bar{\varphi}) = \mathbf{Q}(\varphi)$, which contains a representation whose character is χ with multiplicity one. Hence χ can be realized over $\mathbf{Q}(\varphi)$.

COROLLARY. *Let the integers n_i be as in the proof of Proposition* 2, *and let* $n = \text{g.c.d.}\{n_i\}$. *Then* $\mathbf{Q}(\sqrt[(q^n - 1)]{1})$ *is a splitting field for the p-regular characters of B.*

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