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MAJORIZATION ON A PARTIALLY ORDERED SET

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ABSTRACT. We extend the classical concept of set majorization to the case where the set is partially ordered. We give a useful property which characterizes majorization on a partially ordered set. Quite unexpectedly, the proof of this property relies on a theorem of Shapley on convex games. We also give a theorem which is parallel to the Schur-Ostrowski theorem in comparing two sets of parameters in a function.

1. Introduction. The classical concept of majorization is defined on two n-sets of numbers $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ as follows. Let $a_{[i]}$ and $b_{[i]}$ denote the ith largest numbers in A and B, respectively. Then A is said to *majorize* B if and only if

$$\sum_{i=1}^{k} a_{[i]} > \sum_{i=1}^{k} b_{[i]} \quad \text{for } k = 1, \ldots, n-1$$

and

$$\sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]}.$$

The concept of majorization is closely related to the concept of a Schur function. A function $f(x_1, \ldots, x_n)$ is called a *Schur function* [4] if, for all i and j,

$$\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right)(x_i - x_j) > 0.$$

The following theorem connects the two concepts:

THEOREM 1.1 (SCHUR [5], OSTROWSKI [4]). *$f(a_1, \ldots, a_n) > f(b_1, \ldots, b_n)$ for all A majorizing B if and only if f is a Schur function.*

Set majorization can be naturally extended to vector majorization. We say that an n-vector $A = (a_1, \ldots, a_n)$ majorizes another n-vector $B = (b_1, \ldots, b_n)$ if and only if

$$\sum_{i=1}^{k} a_i > \sum_{i=1}^{k} b_i \quad \text{for } k = 1, \ldots, n-1,$$

and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

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The concept of vector majorization has been proved useful in many instances ([1], [3]) where vector optimization is concerned.

Vector majorization can be interpreted in a different way which leads to a further extension. Let $P = \{p_1, \ldots, p_n\}$ be a set of points ordered linearly by "$\rightarrow$" and let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two sets of weights where a_i and b_i are associated with p_i for $i = 1, \ldots, n$. Then A is said to majorize B if for every point p_i in P, the sum of the a weights of all points $\{p_j: p_j \rightarrow p_i \text{ or } p_j = p_i\}$ is not less than the sum of the b weights on the same set of points. With this viewpoint, it is natural to consider majorization on a set of points P partially ordered by "$\rightarrow$" (read "dominates"). For $P' \subseteq P$ let $A(P')$ (or $B(P')$) denote the sum of the weights (b weights) of all the points $\{p_j: p_j \in P' \text{ or } p_j \rightarrow p_i \text{ for some } p_i \in P'\}$. Then we say that *$A$ majorizes B on P* if $A(P) = B(P)$ and for every $P' \subset P, A(P') \geqslant B(P')$. Note that when P is a linearly ordered set, then A majorizing B on P is reduced to the definition of vector majorization.

In this paper we give a useful property which characterizes majorization on a partially ordered set. It turns out that to prove this property, we need to resort to some concepts and results in characteristic function games. Therefore we give a brief sketch of what we need from characteristic function games in §2. Using the characterization property, we prove a theorem parallel to the Schur-Ostrowski theorem on set majorization. A similar theorem on vector majorization follows as a corollary to our theorem.

2. Some concepts and results in characteristic function games. For a set of players $N = \{1, \ldots, n\}$ a *characteristic function* $v(\cdot)$ is a real valued function assigning to each subset $S \subseteq N$ the number $v(S)$. This number may be thought of as describing the potential worth of the coalition S. The function $v(\cdot)$ completely determines the strategic possibilities of the game. A game is *convex* if its characteristic function satisfies $v(\phi) = 0$ and

$$v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$$

for any $S, T \subseteq N$.

The *core* of a characteristic function game is the set of solutions to the following set of equations

$$\sum_{i \in S} x_i \geqslant v(S) \quad \text{for } S \subset N,$$

$$\sum_{i \in N} x_i = v(N).$$

The core can be described intuitively as the set of payoff vectors that leave no coalition in a position to improve the payoffs of all its members. A characteristic function game need not have a core. However, Shapley [6] proved the following.

THEOREM 2.1. *The core exists for every convex game.*

It is quite unexpected that this theorem will be needed to prove a property of majorization on a partially ordered set.

3. The main theorems. Consider a given set of points $P = \{p_1, \ldots, p_n\}$ partially ordered by "$\rightarrow$" and a set of weights A associated with P. Let p_i and p_j be two points in P such that $p_i \rightarrow p_j$. Then a *flow* from a_i to a_j is a transformation from A to A' where

$$\begin{aligned} a_i' &= a_i - \delta, \\ a_j' &= a_j + \delta, \\ a_k' &= a_k \quad \text{for } k \neq i, j, \end{aligned}$$

for some $\delta > 0$.

THEOREM 3.1. *Let A and B be two sets of weights associated with the partially ordered set P. Then A majorizes B on P if and only if A can be transformed into B by a finite set of flows (in fact, at most $\binom{n}{2}$ flows are needed).*

PROOF. (i) *The "if" direction.* If A can be transformed into A' by a flow, then clearly A majorizes A' on P. Since majorization on P is transitive, A majorizes B on P.

(ii) *The "only if" direction.* Suppose A majorizes B on P; we show that there exists a finite set of flows transforming A into B. We prove this by induction on the number of points in P. Let q be a point of P which is not dominated by any other point of P. If q does not dominate any other point we ignore q and prove Theorem 3.1 by induction. Otherwise let $q_1, \ldots, q_j$ be the points dominated by q, but not dominated by any other points dominated by q. If $a_q = b_q$, then again, we can ignore q and prove Theorem 3.1 by induction. So we assume $a_q - b_q = \theta > 0$. We now show that there exists a set of weights A' which can be obtained from A by flowing the amount $\theta_i \geqslant 0$ from q to q_i, $i = 1, \ldots, j$, such that $\sum_{i=1}^{j} \theta_i = \theta$ and A' majorizes B on P. Once this has been proved, then by induction A' can be transformed into B by a finite set of flows. Consequently, A can be transformed into B by a finite set of flows and Theorem 3.1 follows.

For $P' \subseteq P$, define

$$\bar{A}(P') = A(P') - \theta.$$

Let K be a subset of $J = (1, \ldots, j)$. Define

$$v_K = \operatorname*{Max}_{P_K} \big(B(P_K) - \bar{A}(P_K) \big)$$

where P_K is a subset of $P - \{q\}$ containing K but not any element from $J - K$. Then A' majorizes B if and only if

$$\sum_{i \in K} \theta_i \geqslant v_K \quad \text{for every } K \subseteq J.$$

We can define a characteristic function game on the set of players J by treating $\{v_K : K \subseteq J\}$ as the characteristic function. Then

$$\sum_{i \in K} \theta_i \geqslant v_K \quad \text{for every } K \subseteq J$$

is equivalent to the statement that the core of the game exists. We now prove the core exists by showing that the game is convex.

Let P_K^0 and $P_{K'}^0$ be two subsets of J such that

$$v_K = B(P_K^0) - \bar{A}(P_K^0)$$

and

$$v_{K'} = B(P_{K'}^0) - \bar{A}(P_{K'}^0).$$

Then

$$\begin{aligned} v_{K \cup K'} &\geqslant B(P_K^0 \cup P_{K'}^0) - \bar{A}(P_K^0 \cup P_{K'}^0) \\ &= B(P_K^0) + B(P_{K'}^0) - B(P_K^0 \cap P_{K'}^0) \\ &\quad - \bar{A}(P_K^0) - \bar{A}(P_{K'}^0) + \bar{A}(P_K^0 \cap P_{K'}^0) \\ &= v_K + v_{K'} - \big(B(P_K^0 \cap P_{K'}^0) - \bar{A}(P_K^0 \cap P_{K'}^0)\big) \\ &\geqslant v_K + v_{K'} - v_{K \cap K'}. \end{aligned}$$

Therefore the game is convex and so the core exists by Theorem 2.1.

COROLLARY. *Suppose P is a linearly ordered set. Then a necessary and sufficient condition for A majorizing B is the existence of an $n \times n$ triangular matrix $M = \{m_{ij}\}$ such that $m_{ij} \geqslant 0$, $m_{ij} = 0$ for $i < j$, $\sum_{j=1}^{n} m_{ij} = 1$ and $B = MA$ (by interpreting $m_{ij} a_i$ as the amount of flow from p_i to p_j).*

Note that this corollary is very similar to Theorem 46 of [2] which says that a necessary and sufficient condition for a set A majorizing a set B is the existence of a doubly stochastic matrix M such that $B = MA$.

The following theorem is parallel to the Schur-Ostrowski theorem.

THEOREM 3.2. *Let $f(x_1, \ldots, x_n)$ be a function defined over the domain D. Let $P = (p_1, \ldots, p_n)$ be a set of points partially ordered by "$\to$". Then*

$$f(a_1, \ldots, a_n) \geqslant f(b_1, \ldots, b_n)$$

for all A majorizing B on P if and only if f is such that for every i and j, $p_i \to p_j$ implies

$$\frac{\partial f}{\partial x_i} \geqslant \frac{\partial f}{\partial x_j} \quad \textit{over all } X \in D.$$

PROOF. The "only if" part is trivial. The "if" part can be proved as follows.

If there is a flow transforming A into A', then clearly, $f(A) \geqslant f(A')$. From Theorem 3.1, there exists a finite set of flows transforming A into B. Therefore Theorem 3.2 follows.

COROLLARY. *Let $f(x_1, \ldots, x_n)$ be a function defined over domain D. Let $P = (p_1, \ldots, p_n)$ be a vector. Then*

$$f(a_1, \ldots, a_n) \geqslant f(b_1, \ldots, b_n)$$

for all A majorizing B (in the vector sense) if and only if f is such that

$$\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j} \right)(i - j) \geqslant 0.$$

4. Conclusions. In this paper we extend the classical concept of set majorization to the case where the set is partially ordered. We give a mathematical definition of this new concept which includes "vector majorization" as a special case. Let P and P' be two partial orders on the same set such that $p_i \to p_j$ in P implies $p_i \to p_j$ in P'. Then surprisingly, it is not true that if A majorizes B on the set under P', then A majorizes B on the set under P (nor conversely) as is clear from the following example:

EXAMPLE. Let $P = (p_1 \to p_2, p_1 \to p_3)$, $P' = (p_1 \to p_2 \to p_3)$, $A = (x_1 = .5, x_2 = .5, x_3 = 0)$, $B = (y_1 = .4, y_2 = .3, y_3 = .3)$. Then A majorizes B on P' but not on P.

We also prove a property which characterizes majorization on a partially ordered set. Quite unexpectedly, the proof relies on a theorem of Shapley on convex games. Furthermore, we prove a theorem which is parallel to the Schur-Ostrowski theorem in comparing two functions except that the set majorization condition is replaced by a condition relating to the new notion of majorization.

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