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GROUP RINGS IN WHICH EVERY LEFT IDEAL IS A RIGHT IDEAL

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ABSTRACT. Let $K[G]$ denote the group ring of G over the field K. In this note we characterize those group rings in which all left ideals are right ideals.

Let R be a ring. We say that R is l.i.r.i. if every left ideal is a right ideal. A ring is l.a.r.i. if every left annihilator is a right ideal. Our notation follows that of [2].

The main results are

THEOREM I. *Let K be a field and let G be a nonabelian periodic group. Then if $K[G]$ is l.a.r.i. one of the following occurs*

(i) Char $K = 0$ *and G is a Hamiltonian group such that for each odd exponent, n, of G the quaternion algebra over the field $K(\xi_n)$, where ξ_n is a primitive nth root of unity, is a division ring.*

(ii) Char $K = 2$ *and K does not contain any primitive cube root of unity. Moreover $G \cong Q \times A$, where Q is the quaternion group of order* 8 *and A is abelian in which each element has odd order and if n is an exponent for A, the least integer $m > 1$ satisfying $2^m \equiv 1 \pmod{n}$ is odd.*

Conversely if $K[G]$ satisfies either (i) *or* (ii), *then $K[G]$ is l.i.r.i. and, in particular, it is l.a.r.i.*

Observe that if char $K > 2$ and G is periodic, then $K[G]$ is l.a.r.i. if and only if G is abelian.

THEOREM II. *Let $K[G]$ denote the group ring over a nonabelian group. Then the following are equivalent*

(i) *$K[G]$ is l.i.r.i.*

(ii) *G is locally finite and if $\alpha,\beta \in K[G]$ with $\alpha\beta = 0$, then $\beta\alpha = 0$.*

(iii) *G is locally finite and $K[G]$ is l.a.r.i.*

If we combine the above theorems we get necessary and sufficient conditions for $K[G]$ to be l.i.r.i.

By using the antiautomorphism of $K[G]$ given by $\sum_{x\in G} k_x x \mapsto \sum_{x\in G} k_x x^{-1}$ we see that $K[G]$ is l.i.r.i. (l.a.r.i.) if and only if $K[G]$ is r.i.l.i. (r.a.l.i.).

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LEMMA 1. (i) *$K[G]$ is l.i.r.i. if and only if for every finitely generated subgroup $H \subseteq G$, $K[H]$ is l.i.r.i.* (ii) *If $K[G]$ is l.i.r.i., then all subgroups of G are normal.* (iii) *Suppose that G is periodic. If $K[G]$ is l.a.r.i., then all subgroups of G are normal.*

PROOF. (i) First we suppose that for every finitely generated subgroup $H \subseteq G$, $K[H]$ is l.i.r.i. Let $I \subseteq K[G]$ be a left ideal. Let $\alpha \in I$, $g \in G$. We set $H = \langle g, \operatorname{Supp} \alpha\rangle$. Then $I \cap K[H]$ is a left ideal of $K[H]$ and hence $I \cap K[H]$ is an ideal of $K[H]$, since H is finitely generated. Now $g \in H$ and $\alpha \in I \cap K[H]$ so $\alpha g \in I \cap K[H] \subseteq I$. Therefore we have shown that $Ig \subseteq I$ for any $g \in G$ and so I is a right ideal. Conversely let H be a subgroup of G and suppose that $I \subseteq K[H]$ is a left ideal of $K[H]$. Let $\{x_i\}$ be a set of left coset representatives for H in G. Then $K[G]$ is a free right $K[H]$-module with basis $\{x_i\}$. Thus we have $K[G] = \dot{\Sigma} x_i K[H]$. Denote $\dot{\Sigma} x_i I$ by J. Clearly J is a left ideal of $K[G]$. If we suppose that $K[G]$ is l.i.r.i., then we have that J is a right ideal of $K[G]$. Let $h \in H$. Then

$$Ih \subseteq Jh \cap K[H] \subseteq J \cap K[H] = I$$

and so I is a right ideal of $K[H]$.

(ii) In order to prove that all subgroups of G are normal it suffices to see that all cyclic subgroups are normal. Let $a,g \in G$. Consider the left ideal $I = K[G](1 - a)$. Then I is an ideal, since $K[G]$ is l.i.r.i. Thus $g^{-1}(1 - a)g \in I$ and $1 - g^{-1}ag = \alpha(1 - a)$ for a suitable element $\alpha \in K[G]$. Now we use the $K[\langle a\rangle]$-homomorphism θ: $K[G] \to K[G]$ in which $\Sigma_{x \in G} k_x x \mapsto \Sigma_{x \in \langle a\rangle} k_x x$ and we obtain $1 - \theta(g^{-1}ag) = \theta(\alpha)(1 - a)$. Since $1 - a$ is not invertible we have that $\theta(g^{-1}ag) \neq 0$. Hence $g^{-1}ag \in \langle a\rangle$.

(iii) Suppose that G is periodic and $K[G]$ is l.a.r.i. Let $g \in G$. Set $H = \langle g\rangle$. Lemma 1.2 [2, Chapter 3] yields that $l(\hat{H}) = K[G]\omega(K[H])$. On the other hand we have that $H = \{x \in G\colon x - 1 \in K[G]\omega(K[H])\}$. By hypothesis $l(\hat{H})$ is an ideal, then it is easy to see that H is normal in G.

We recall that a nonabelian group G such that all subgroups are normal is a Hamiltonian group, that is [1, Theorem 12.5.4]

$$G = Q \times A \times B$$

where Q is the quaternion group of 8 elements, A is an abelian group such that every element has odd order, and B is an abelian group of exponent 2. For the rest of this paper we fix this notation.

LEMMA 2. *Suppose that $K[G]$ is l.a.r.i. Let $\alpha,\beta \in K[G]$ such that $\alpha\beta = 0$. Then $\beta\alpha = 0$.*

PROOF. Suppose I and J are ideals of $K[G]$ with $IJ = 0$. Then JI is an ideal of $K[G]$ and $\operatorname{tr}(JI) = \operatorname{tr}(IJ) = 0$. Thus $JI = 0$ since any ideal of trace zero is zero. Now let $K[G]$ be l.a.r.i. If $\alpha,\beta \in K[G]$ with $\alpha\beta = 0$, then $IJ = 0$ where $I = K[G]\alpha K[G]$ and $J = K[G]\beta K[G]$. Thus $JI = 0$ and hence $\beta\alpha = 0$.

In characteristic 2 we need the following

LEMMA 3. *Let K be a field of characteristic 2. Suppose that K does not contain any primitive cube root of unity. Put*

$$Q = \langle a,b \mid a^2 = b^2, a^4 = 1, b^{-1}ab = a^{-1}\rangle.$$

Then if $\alpha = \Sigma k_x x \in K[\langle a\rangle]$ such that $|\alpha| = 1$ (where $|\alpha| = \Sigma k_x$) we have

$$1 + (\alpha b)^2 = (1 + a^2)u$$

where $u \in K[\langle a\rangle]$ is a unit.

PROOF. Let $\alpha = k_1 + k_2 a + k_3 a^2 + k_4 a^3 \in K[\langle a\rangle]$ with $\Sigma k_i = 1$. Then a calculation proves that

$$1 + (\alpha b)^2 = (1 + a^2)(1 + (k_1 + k_3)(k_2 + k_4)a).$$

Since Q is a 2-group and char $K = 2$ we know that $K[Q]$ is a local ring whose maximal ideal is $\{\alpha \in K[Q]: |\alpha| = 0\}$. Suppose by contradiction that $1 + (k_1 + k_3)(k_2 + k_4)a$ is not a unit. Then $(k_1 + k_3)(k_2 + k_4) = 1$, and since $\Sigma k_i = 1$ we see that $k_1 + k_3$ is a primitive cube root of unity. Since K does not contain any primitive cube root of unity we have a contradiction.

THE PROOF OF THEOREM I. Suppose that G is a nonabelian periodic group and $K[G]$ is l.a.r.i. Then Lemma 1(iii) yields that $G = Q \times A \times B$. First we observe that the case char $K > 2$ is not possible. Since $K[G]$ is l.a.r.i. clearly $K[Q]$ so is. But in char > 2 we have

$$K[Q] \cong K \dotplus K \dotplus K \dotplus K \dotplus M(2, K)$$

and this is a contradiction, since $M(2, K)$ is not l.a.r.i. Suppose char $K = 0$. Let n be an exponent for A and let $x \in A$ such that $o(x) = n$. Then $K[\langle x\rangle]$ is a direct sum of fields

$$K[\langle x\rangle] \cong K(\xi_n) \dotplus L_1 \dotplus \cdots \dotplus L_m$$

where $o(\xi_n) = n$. On the other hand we have

$$K[Q] \cong K \dotplus K \dotplus K \dotplus K \dotplus ((-1, -1)/K)$$

where the last summand is the quaternion algebra over K. Since $K[Q \times \langle x\rangle] \cong K[Q] \otimes_K K[\langle x\rangle]$ we get that $((-1, -1)/K) \otimes K(\xi_n) \cong ((-1, -1)/K(\xi_n))$ is a direct summand of $K[Q \times \langle x\rangle]$ and so $((-1, -1)/K(\xi_n))$ is l.a.r.i. Therefore the quaternion algebra over $K(\xi_n)$ is a division ring. Conversely suppose that $K[G]$ satisfies (i). Then we will prove that $K[G]$ is l.i.r.i. It follows from Lemma 1(i) that it suffices to consider G finite. Then

$$G \cong Q \times A \times (Z/2Z) \times \cdots \times (Z/2Z)$$

(m copies of $Z/2Z$) and we get

$$K[G] \cong K[Q \times A] \dotplus \cdots \dotplus K[Q \times A]$$

(2^m copies of $K[Q \times A]$). Clearly we can suppose that $G = Q \times A$. Then it is easy to see that

$$K[G] \cong K[A] \dotplus K[A] \dotplus K[A] \dotplus K[A] \dotplus \prod_i \left(\frac{-1, -1}{K(\xi_i)}\right)$$

where $o(\xi_i)$ are exponents for A. Hence we see that $K[G]$ is a product of l.i.r.i. rings. Therefore $K[G]$ is l.i.r.i.

Char $K = 2$. First we observe that if K contains a primitive cube root of unity then $K[G]$ is not l.a.r.i. From Lemma 2 it suffices to exhibit elements $\alpha, \beta \in K[G]$ such that $\alpha\beta = 0$ but $\beta\alpha \neq 0$. If ξ is a primitive cube root of unity we set

$$\alpha = (1 + \xi(1 + \xi a)b),$$
$$\beta = (1 + \xi(1 + \xi a)b)(1 + a)b.$$

A calculation proves that $\alpha\beta = 0$ but $\beta\alpha \neq 0$. We now prove that $G = Q \times A$. If this is not the case there exists an element $x \in G - Q$ of order 2 which centralizes G. Again there exist elements

$$\alpha = 1 + (a + b + ab)x,$$
$$\beta = (a + b + ab)(1 + a) + (1 + a)x,$$

such that $\alpha\beta = 0$ but $\beta\alpha \neq 0$ and so $K[G]$ is not l.a.r.i. Let n be an exponent for A and $x \in A$ such that $o(x) = n$. Since char $K = 2$ we have that $K[\langle x \rangle]$ is semisimple, and so

$$K[\langle x \rangle] = K(\xi_n) \dotplus \cdots \dotplus L_m \text{ where } o(\xi_n) = n.$$

Then $K[Q] \otimes K(\xi_n) \cong K(\xi_n)[Q]$ is a direct factor of $K[Q \times \langle x \rangle]$. By hypothesis $K(\xi_n)[Q]$ is l.a.r.i. By above $K(\xi_n)$ does not contain any primitive cube root of unity. Therefore $2 \nmid m$, where m is the degree of the extension $(Z/2Z(\xi_n))/(Z/2Z)$. But m is precisely the least integer satisfying $2^m \equiv 1 \pmod{n}$. Conversely suppose that $K[G]$ satisfies (ii). We shall prove that $K[G]$ is l.i.r.i. Again from Lemma 1(i) we can consider that G is finite. Then

$$K[A] \cong K(\xi_i) \dotplus \cdots \dotplus K(\xi_m)$$

and so

$$K[Q \times A] \cong K(\xi_1)[Q] \dotplus \cdots \dotplus K(\xi_m)[Q].$$

By hypothesis the field $K(\xi_i)$ does not contain any primitive cube root of unity. Since a product of l.i.r.i. rings is l.i.r.i., we have only to prove that if a field K does not contain any primitive cube root of unity, then $K[Q]$ is l.i.r.i. Let $I \subseteq K[G]$ be a left ideal. Suppose that $\alpha \in I$. We can write α in the form $\alpha = \alpha_1 + \alpha_2 b$, where $\alpha_i \in K[\langle a \rangle]$. The first task is to show that $\alpha_i(1 + a^2) \in I$. Note that if $\alpha_1(1 + a^2) \in I$, then, since $1 + a^2$ is central, $\alpha_2 b(1 + a^2) \in I$. Again $\alpha_2(1 + a^2)$ is central and therefore $b\alpha_2(1 + a^2) \in I$. Since I is a left ideal $\alpha_2(1 + a^2) \in I$. Thus we need only to prove that $\alpha_1(1 + a^2) \in I$. If α is a unit, then $I = K[Q]$. Thus we may suppose that α is not a unit. Then we have $|\alpha_1| + |\alpha_2| = 0$. Suppose that α_1 is a unit. Then $1 + \alpha_1^{-1}\alpha_2 b \in I$. Clearly $1 + (\alpha_1^{-1}\alpha_2 b)^2 \in I$, so Lemma 3 yields that $1 + a^2 \in I$. Hence $\alpha_1(1 + a^2) \in I$. If α_1 is not a unit, then we have $|\alpha_1| = 0$ and hence $|\alpha_2| = 0$. Therefore $\alpha_1 = \beta_1(1 + a)$ and $\alpha_2 = \beta_2(1 + a)$ for suitable elements $\beta_i \in K[\langle a \rangle]$. Thus $\alpha = (\beta_1 + \beta_2 ab)(1 + a)$. If $\beta_1 + \beta_2 ab$ is a unit we obtain that $1 + a \in I$ and so $\alpha_1(1 + a^2) = \alpha_1(1 + a)^2 \in I$. Hence we may consider that $|\beta_1| + |\beta_2| = 0$. If β_1 is a unit, then $(1 + \beta_1^{-1}\beta_2 ab)(1 + a) \in I$. Again we use Lemma 3 and we get that $(1 + a^2)(1 + a) \in I$. Thus $\alpha_1(1 + a^2) = \beta_1(1 + a)(1 + a^2) \in I$. Finally if β_1 is not a unit we have $\beta_1 = \gamma_1(1 + a)$ for certain $\gamma_1 \in K[\langle a\rangle]$. Therefore $\alpha_1(1 + a^2) = \gamma_1(1 + a^2)(1 + a^2) = 0$ and, certainly, $\alpha_1(1 + a^2) \in I$. Now we will prove that $\alpha x \in I$ for any $x \in Q$. Since $Q = \langle a, b\rangle$ it suffices to see that $\alpha a, \alpha b \in I$. By using the automorphism of Q given by $a \mapsto b$, $b \mapsto a$ we see that we have only to prove that $\alpha a \in I$. $\alpha_2(1 + a^2)$ is central and so

$$\alpha a = \alpha_1 a + \alpha_2 ba = a\alpha + ab\alpha_2(1 + a^2).$$

Since $a\alpha \in I$ and by above $\alpha_2(1 + a^2) \in I$, the result follows.

THE PROOF OF THEOREM II. (i) $\Rightarrow$ (ii). It follows from Lemma 1(ii) that all subgroups of G are normal. Since G is not abelian, it is a Hamiltonian group and, clearly, locally finite. Since if a ring is l.i.r.i., then it is l.a.r.i. Lemma 2 completes the proof. Trivially (ii) implies (iii). It follows from Theorem I that (iii) implies (i). The result follows.

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