ON THE RIGIDITY OF GRADED ALGEBRAS¹

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ABSTRACT. If $\mathcal{G}$ is a graded algebra (separated and complete) over a field of characteristic zero and $\mathcal{G}$ is rigid in the category of algebras, then $\mathcal{G}$ is rigid in the category of filtered algebras.

1. Introduction. If $\mathcal{G}$ is a graded algebra (separated and complete) over a field $\mathcal{K}$ of characteristic zero and $\mathcal{G}$ is rigid in the category of algebras, then, as we show here, $\mathcal{G}$ is rigid in the category of filtered algebras and hence isomorphic, as a filtered algebra, to any $\mathcal{F}$ whose (complete) associated graded algebra is $\mathcal{G}$. Thus if a graded algebra is rigid in its operational structure, then it must also be rigid in its filtration structure.

The restriction that the graded algebra under consideration be an algebra over a field of characteristic zero (or more precisely, an algebra over a commutative unital coefficient ring $\mathcal{K}$ which is itself an algebra over $\mathbb{Q}$) is required for the proof of Theorem 2 which we give here. The result that in a trivial deformation the infinitesimal of the deformation is a coboundary is critical to the proof and the following examples due to M. Gerstenhaber ([2]–[5]) show that this result need not hold for algebras defined over a field of characteristic $p > 0$.

Let $\mathcal{K}$ be a field of characteristic 2 and the algebra $A = \mathcal{K}[x]/x^2$. Consider the deformation $A_t = \mathcal{K}[x][[t]]/(x^2 + t)$. The multiplication on A_t is $f_t = \pi + tF$ where $F(1, 1) = F(1, x) = F(x, 1) = 0$, $F(x, x) = 1$ and π is the original multiplication on A . A_t is a nontrivial deformation of A since there is no power series of the form $w = x + a_1t + a_2t^2 + \dots$ with coefficients $a_1, a_2, \dots$ in A such that with the multiplication in A , $w^2 \equiv t \pmod{t^2}$. Thus F is not a coboundary since if it were, A_t would be a trivial deformation. Now consider another deformation of A , $B_t = \mathcal{K}[x][[t]]/(x^2 + t^2)$. The multiplication on B_t is defined by $g_t = \pi + tF$ where F is the same as defined above. B_t is a trivial deformation of A under the mapping $\Phi_t = x + t$ which is an automorphism since $\mathcal{K}$ is a field of characteristic 2. Thus B_t is a trivial deformation but the infinitesimal of this deformation, F , is not a coboundary.

Given the history of results in algebraic deformation theory it is unlikely

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that the major result of this paper translates exactly to algebras over fields of characteristic $p > 0$.

The algebra $\mathcal{G}$ under consideration need not be associative. It is sufficient that $\mathcal{G}$ be an element of a category in which the $\mathcal{K}$ -split extensions of $\mathcal{G}$ by $\mathcal{G}$ considered as a $\mathcal{G}$ -bimodule form a group $H^2(\mathcal{G}, \mathcal{G})$. The list of such categories includes associative, Lie, commutative associative, and nilpotent associative.

2. Gerstenhaber [2] showed that given a complete, separated filtered ring $\mathcal{F}$ there exists a one parameter family of deformations $f_t = \pi + tF_1 + t^2F_2 + t^3F_3 + \dots$ of the (complete) associated graded ring $\mathcal{G}$ and $\mathcal{G}$ with multiplication f_1 is isomorphic to $\mathcal{F}$ as a filtered ring. Thus $\mathcal{F}$ is a deformation of $\mathcal{G}$ under f_t and the infinitesimal of this deformation is F_1 .

There is an alternate way of viewing $\mathcal{F}$ as a deformation of its associated graded ring $\mathcal{G}$. Following Gerstenhaber [3], we consider

$$\text{App } \mathcal{F} = \left\{ \sum_{i=-n}^{\infty} a_i t^i \mid a_i \in F^i \mathcal{F} \right\} \subseteq \mathcal{F}((t)).$$

Let

$$F^n \text{App } \mathcal{F} = \left\{ \sum_{i=-n}^{\infty} a_i t^i \mid a_i \in F^{i+n} \mathcal{F} \right\}$$

and let $\text{App}_n \mathcal{F} = \text{App } \mathcal{F} / F^{n+1} \text{App } \mathcal{F}$; $\text{App}_0 \mathcal{F} = \mathcal{G}$. There is a natural inclusion from $F^{n+1} \mathcal{F} / F^{n+2} \mathcal{F} \rightarrow F^n \mathcal{F} / F^{n+2} \mathcal{F}$. The direct sum over all n of these maps yields:

$$0 \rightarrow \bigoplus F^{n+1} \mathcal{F} / F^{n+2} \mathcal{F} \rightarrow \text{App}_1 \mathcal{F}$$

where $\bigoplus F^{n+1} \mathcal{F} / F^{n+2} \mathcal{F}$ is $\mathcal{G}$ with the degree shifted by one and zero multiplication. Denote this subobject of $\text{App}_1 \mathcal{F}$ by $\text{sh } \mathcal{G}$. Thus there exists an exact sequence: $0 \rightarrow \text{sh } \mathcal{G} \rightarrow \text{App}_1 \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ which is a singular extension and therefore an element of $\text{Baer}^2(\mathcal{G}, \text{sh } \mathcal{G})$.

THEOREM 1. *The singular extension: $0 \rightarrow \text{sh } \mathcal{G} \rightarrow \text{App}_1 \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ corresponds with the classical infinitesimal of the deformation from $\mathcal{G}$ to $\mathcal{F}$.*

PROOF. Let $f_t = \pi + tF_1 + t^2F_2 + \dots$ be a one parameter family of deformations from $\mathcal{G}$ to $\mathcal{F}$. The n th graded part of $\text{App}_1 \mathcal{F}$ is

$$\{a_n t^n + a_{n+1} t^{n+1} \mid a_n \in F^n \mathcal{F}, a_{n+1} \in F^{n+1} \mathcal{F}\}.$$

The multiplication in $\text{App}_1 \mathcal{F}$ is given by

$$\begin{aligned} f_t(a_i t^i + a_{i+1} t^{i+1}, b_j t^j + b_{j+1} t^{j+1}) \\ = a_i b_j t^{i+j} + (a_i b_{j+1} + a_{i+1} b_j) t^{i+j+1} + t^{i+j+1} F_1(a_i, b_j). \end{aligned}$$

Denote the elements of $\text{App}_1 \mathcal{F}$ by $a + bt$. Define an additive section s of the singular extension: $0 \rightarrow \text{sh } \mathcal{G} \rightarrow \text{App}_1 \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ by $s(a) = a + 0 \cdot t$. Then

$$f_t(s(a), s(b)) - s(a \cdot b) = a \cdot b + tF_1(a, b) - a \cdot b = tF_1(a, b)$$

which is the classical infinitesimal of the deformation f_t .

We may define

$$\mathrm{sh}^n \mathcal{G} = \bigoplus_i F^{i+n} \mathcal{F} / F^{i+n+1} \mathcal{F} \quad \text{and} \quad \mathrm{sh} \mathrm{App}_n = \bigoplus_i F^{i+1} / F^{i+n+2}.$$

The exact sequence:

$$0 \rightarrow \mathrm{sh}^2 \mathcal{G} \rightarrow \mathrm{sh} \mathrm{App}_1 \mathcal{F} \rightarrow \mathrm{App}_1 \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0 \quad (1)$$

is trivial with solution $E_2: 0 \rightarrow \mathrm{sh} \mathrm{App}_1 \mathcal{F} \rightarrow \mathrm{App}_2 \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$. If the one parameter family of deformations from $\mathcal{G}$ to $\mathcal{F}$ is defined by $g_i = \pi + t^2 F_2 + t^3 F_3 + \dots$, then $\mathrm{App}_1 \mathcal{F}$ and $\mathrm{App}_1 \mathcal{G}$ are isomorphic as filtered rings and therefore $E'_2: 0 \rightarrow \mathrm{sh} \mathrm{App}_1 \mathcal{F} \rightarrow \mathrm{App}_2 \mathcal{G} \rightarrow \mathcal{G} \rightarrow 0$ is also a solution of (1). The equivalence classes of solutions form a principal homogeneous space over $\mathrm{Baer}^2(\mathcal{G}, \mathrm{sh}^2 \mathcal{G})$. Thus the difference $E_2 - E'_2$ is an element of $\mathrm{Baer}^2(\mathcal{G}, \mathrm{sh}^2 \mathcal{G})$ and is equivalent to:

$$0 \rightarrow \mathrm{sh}^2 \mathcal{G} \rightarrow \mathrm{App}_2 \mathcal{F} \times_{\mathrm{App}_1 \mathcal{F}} \mathrm{App}_2 \mathcal{G} / \mathrm{diag}(\mathrm{sh} \mathrm{App}_1 \mathcal{F}) \rightarrow \mathcal{G} \rightarrow 0.$$

This singular extension corresponds to F_2 . In general, if the deformation from $\mathcal{G}$ to $\mathcal{F}$ is defined by $\pi + t^n F_n t^{n+1} F_{n+1} + \dots$ for $n > 1$, then the singular extension

$$0 \rightarrow \mathrm{sh}^n \mathcal{G} \rightarrow \mathrm{App}_n \mathcal{F} \times_{\mathrm{App}_{n-1} \mathcal{F}} \mathrm{App}_n \mathcal{G} / \mathrm{diag}(\mathrm{sh} \mathrm{App}_{n-1} \mathcal{F}) \rightarrow \mathcal{G} \rightarrow 0$$

corresponds to F_n .

The following theorem is the major result of this paper.

THEOREM 2. *If $\mathcal{G}$ is a graded algebra (separated and complete) over a field $\mathcal{K}$ of characteristic zero and $\mathcal{G}$ is rigid in the category of algebras, then $\mathcal{G}$ is rigid in the category of filtered algebras.*

PROOF. Let $\mathcal{F}$ be a filtered algebra over a field $\mathcal{K}$ of characteristic zero whose (complete) associated graded algebra is $\mathcal{G}$. Let $f_i = \pi + t F_1 + t^2 F_2 + \dots$ define a one parameter family of deformations from $\mathcal{G}$ to $\mathcal{F}$ in the category of filtered algebras. We shall show that f_i is trivial, i.e. that there exists a filtration preserving algebra isomorphism Φ_i such that $\Phi_i \pi(a, b) = f_i(\Phi_i a, \Phi_i b)$. Since $\mathcal{G}$ is an algebra over a field of characteristic zero and $\mathcal{G}$ is rigid in the category of algebras, F_1 , the infinitesimal of the deformation, must be a coboundary (Gerstenhaber [2], [5]). By Theorem 1, the singular exact sequence:

$$0 \rightarrow \mathrm{sh} \mathcal{G} \rightarrow \mathrm{App}_1 \mathcal{F} \xrightarrow{p} \mathcal{G} \rightarrow 0 \quad (2)$$

corresponds to F_1 . Since F_1 is a coboundary there is a splitting map $s: \mathcal{G} \rightarrow \mathrm{App}_1 \mathcal{F}$ which does not necessarily preserve filtration. We have $s(a_i) = l_i + b_i + h_i$ where $a_i \in \mathcal{G}_i$, $l_i = \bigoplus_{k < i} b_k$, $h_i = \bigoplus_{k > i} b_k$, $b_i \in F^i \mathcal{F}$. Since p is gradation-preserving and linear, l_i and h_i are in the kernel of p . If we define $s_0: \mathcal{G} \rightarrow \mathrm{App}_1 \mathcal{F}$ by $s_0(a_i) = b_i$, then s_0 preserves filtration and is additive.

Consider

$$\begin{aligned} s(a_i) \cdot s(a_j) &= (l_i + b_i + h_i) \cdot (l_j + b_j + h_j) \\ &= l_i b_j + b_i l_j + b_i b_j + b_i h_j + h_i b_j \end{aligned} \quad (3)$$

with the remaining terms equal to zero since (2) is a singular extension. We have $s(a_i \cdot a_j) = s(a_i) \cdot s(a_j)$ and thus $s_0(a_i a_j) = (i + j)$ th filtered part of $s(a_i) \cdot s(a_j) = b_i b_j$ by (3). Therefore $s_0(a_i \cdot a_j) = s_0(a_i) \cdot s_0(a_j)$. Also $ps_0(a_i) = i$ th graded parts of $ps(a_i)$ which is a_i . Thus sequence (2) splits in the category of filtered algebras and $F_1 = -\delta\phi_1$ for some $\phi_1 \in C^1(\mathcal{G}, \mathcal{G})$ where ϕ_1 preserves filtration. Set $\Phi_t(a) = a + t\phi_1(a)$. Then

$$f_t(\Phi_t a, \Phi_t b) = \Phi_t(\pi(a, b) + t^2 F_2(a, b) + t^3 F_3(a, b) + \dots).$$

By the remarks following Theorem 1, F_2 corresponds to an element of $\text{Baer}^2(\mathcal{G}, \text{sh}^2 \mathcal{G})$. The same argument as above shows that this sequence splits in the category of filtered algebras and therefore $\pi + t^2 F_2 + t^3 F_3 + \dots$ is equivalent to $\pi + t^3 F_3 + \dots$ in the category of filtered algebras. Induction and the observation that F_n corresponds to an element of $\text{Baer}^2(\mathcal{G}, \text{sh}^n \mathcal{G})$ when the deformation is $\pi + t^n F_n + t^{n+1} F_{n+1} + \dots$ yields the desired result. Thus we have shown that if a graded algebra is rigid in its operational structure (i.e. rigid as an algebra) then it must also be rigid in its filtration structure (i.e., rigid as a filtered algebra).

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