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THE STRONGLY PRIME RADICAL[1]

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ABSTRACT. Let R denote a strongly prime ring. An explicit construction is given of the radical in R-mod corresponding to the unique maximal proper torsion theory. This radical is characterized in two other ways analogous to known descriptions of the prime radical in rings. If R is a left Ore domain the radical of a module coincides with the torsion submodule.

1. The strongly prime radical. The terminology of radicals in modules is that of Stenstrom [3]. Throughout this paper all rings have a unity and all modules are unital left modules. For a ring R the category of R-modules is denoted by R-mod.

A functor σ: $R\text{-mod} \to R\text{-mod}$ is called a *preradical* if $\sigma(M)$ is a submodule of M and $\sigma(M)\alpha \subseteq \sigma(N)$ for each morphism $M \xrightarrow{\alpha} N$ in R-mod. A preradical α is called a *radical* if $\sigma(M/\sigma(M)) = 0$ for all $M \in R$-mod. A preradical σ is called *left exact* if $\sigma(N) = N \cap \sigma(M)$ whenever $N \subseteq M$ in R-mod (equivalently, if σ is a left exact functor). One method of constructing left exact radicals is given by the following result.

PROPOSITION 1. *Let $\mathfrak{M}$ be any nonempty class of modules closed under isomorphisms. For any module M define*

$$\sigma(M) = \bigcap \{K \mid K \subseteq M, M/K \in \mathfrak{M}\}.$$

It is assumed that $\sigma(M) = M$ if $M/K \notin \mathfrak{M}$ for all $K \subseteq M$. Then

(1) $\sigma[M/\sigma(M)] = 0$ *for all modules M*;

(2) *if $\mathfrak{M}$ is closed under taking nonzero submodules, σ is a radical*;

(3) *if $\mathfrak{M}$ is closed under taking essential extensions, then $\sigma(M) \cap N \subseteq \sigma(N)$ for all submodules $N \subseteq M$.*

In particular, σ is a left exact radical if $\mathfrak{M}$ is closed under nonzero submodules and essential extensions.

PROOF. The proofs of (1) and (2) are straightforward and so are omitted; the last sentence follows from (2) and (3). To prove (3) let $N \subseteq M$ be modules. We must verify that $N \cap \sigma(M) \subseteq K$ whenever $N/K \in \mathfrak{M}$. By Zorn's lemma, choose W maximal in

$$\mathcal{S} = \{W \mid K \subseteq W \subseteq M, W \cap N = K\}.$$

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We claim that $(W + N)/W$ is essential in M/W. For if

$$X/W \cap (W+N)/W = 0$$

where $X/W \neq 0$ then $X \supset W$ so $X \cap N \supset K$ by the choice of W. Suppose $x \in (X \cap N) - K$. Then

$$x + W \in X/W \cap (W+N)/W = 0$$

so $x \in N \cap W = K$, a contradiction. Hence $(W + N)/W \subseteq M/W$ is essential. Since

$$(W+N)/W \cong N/(W \cap N) = N/K \in \mathfrak{M},$$

it follows that $M/W \in \mathfrak{M}$ and so $\sigma(M) \subseteq W$. Thus $\sigma(M) \cap N \subseteq W \cap N = K$ as required. □

We are going to apply this to the following class of modules: An R-module M is called *strongly prime* [1] if $M \neq 0$ and, for each nonzero element $m \in M$, there exists a finite subset $\{r_1, \ldots, r_k\} \subseteq R$ (depending on m) such that $rr_im = 0$ for all i ($r \in R$) implies $r = 0$. In [1] the set $\{r_1, r_2, \ldots, r_k\}$ is called an *insulator* for m. A ring R is called *left strongly prime* if ${}_RR$ is strongly prime (this is not left-right symmetric [1, p. 212]).

PROPOSITION 2. *The class of strongly prime modules is closed under taking isomorphic images, (nonzero) submodules and essential extensions.*

PROOF. It is obviously closed under isomorphic images and nonzero submodules. If $M \subseteq X$ is an essential extension and M is strongly prime let $0 \neq x \in X$. Then $Rx \cap M \neq 0$, say $0 \neq rx \in M$, $r \in R$. Then if $\{r_1, \ldots, r_k\}$ is an insulator for rx it is clear that $\{r_1r, \ldots, r_kr\}$ is an insulator for x. □

Now define the *strongly prime radical* β on R-mod by

$$\beta(M) = \bigcap\{K \mid K \subseteq M, M/K \text{ strongly prime}\},$$

where we assume that $\beta(M) = M$ whenever M has no strongly prime images. Observe that every strongly prime module is faithful. If M is strongly prime and $0 \neq r \in R$ then $rM \neq 0$, say $rm \neq 0$, $m \in M$. If $\{r_1, r_2, \ldots, r_k\}$ is an insulator for rm then it is also an insulator for r in ${}_RR$. It follows that *R is left strongly prime if and only if it has a strongly prime module* [1, p. 220]. In particular, $\beta(M) = M$ for all $M \in R$-mod unless R is a strongly prime ring. In this case Propositions 1 and 2 give:

PROPOSITION 3. *If R is left strongly prime then β is a left exact preradical on* R-mod.

If σ and ρ are two preradicals on R-mod, we say that ρ is *larger* than σ (written $\rho > \sigma$) if $\rho(M) \supseteq \sigma(M)$ for every module $M \in R$-mod. Then we have:

THEOREM 1. *Let R be left strongly prime. Then $\beta(R) = 0$ and $\beta > \sigma$ for every left exact preradical σ on* R-mod *such that $\sigma(R) = 0$.*

PROOF. Clearly $\beta(R) = 0$ (since ${}_RR$ is strongly prime). Suppose σ is a preradical on R-mod for which $\sigma(R) = 0$. Given $M \in R$-mod we must show $\sigma(M) \subseteq \beta(M)$. If not then $\sigma(M) \not\subseteq K$ for some $K \subseteq M$ with M/K strongly prime. If $\alpha\colon M \to M/K$ is the natural map then

$$0 \neq [\sigma(M) + K]/K = \sigma(M)\alpha \subseteq \sigma(M/K)$$

so it suffices to show that $\sigma(M) = 0$ whenever M is strongly prime. Suppose on the contrary that $0 \neq m \in \sigma(M)$. Let $\{r_1, \ldots, r_k\}$ be an insulator for m and define $\lambda\colon R \to M^k$ by $r\lambda = (rr_1m, \ldots, rr_km)$. This is an R-monomorphism and $R\lambda \subseteq \sigma(M)^k \subseteq \sigma(M^k)$. But σ is left exact so $\sigma(R) = 0$ implies

$$0 = \sigma(R\lambda) = R\lambda \cap \sigma(M^k) = R\lambda,$$

a contradiction. $\square$

A nonempty class $\mathfrak{T}$ of modules is called a *pretorsion class* if it is closed under quotients and direct sums; if in addition $\mathfrak{T}$ has the property that M/K, $K \in \mathfrak{T}$ imply $M \in \mathfrak{T}$, then $\mathfrak{T}$ is called a *torsion class*. A pretorsion class is called *hereditary* if it is closed under taking submodules. If a preradical σ on R-mod is given, the class $\mathfrak{T}_\sigma = \{M \mid \sigma(M) = M\}$ is known to be a pretorsion class and the assignment $\sigma \leftrightarrow \mathfrak{T}_\sigma$ is a bijection between left exact preradicals and hereditary pretorsion classes [**3**, p. 138] under which left exact radicals correspond with hereditary torsion classes [**3**, p. 139]. In particular, if R is left strongly prime and β is the strongly prime radical on R-mod, then Theorem 1 implies that $\mathfrak{T}_\beta$ is the torsion class of the largest hereditary torsion theory [**3**, p. 141] on R-mod for which R is torsion-free. The existence of a unique maximal proper torsion theory on R-mod was given in [**1**, p. 220].

2. Further characterizations of the strongly prime radical. In this section we present two characterizations of the strongly prime radical which are analogs of well-known descriptions of the prime radical of a ring. The first gives a generalization of the notion of an m-system. A subset X of an R-module M is called an *fm-system* if $X \neq \varnothing$ and for each $x \in X$ there is a finite subset $F \subseteq R$ (depending on x) such that $rFx \cap X \neq \varnothing$ for all $0 \neq r \in R$.

PROPOSITION 4. *If $N \subseteq M$ are modules then M/N is strongly prime if and only if $M - N$ is an fm-system.*

PROOF. If $M - N$ is an *fm*-system then $M/N \neq 0$ and, if $m \notin N$ for some $m \in M$, choose $F = \{r_1, \ldots, r_k\} \subseteq R$ such that $rFm \cap (M - N) \neq \varnothing$ for all $0 \neq r \in R$. Then F is an insulator for $m + N$. For the converse, reverse the argument. $\square$

One immediate consequence of this proposition is that subdirect products of strongly prime modules are strongly prime. Alternatively, if $K_i \subseteq M$, $i \in I$, are submodules such that M/K_i is strongly prime for each $i \in I$, then $M/\bigcap K_i$ is strongly prime. This follows since $M - \bigcap K_i = \bigcup (M - K_i)$ and the union of a collection of *fm*-systems is again an *fm*-system. In particular, either $\beta(M) = M$ or $M/\beta(M)$ is strongly prime for every module M.

THEOREM 2. *Let R be a left strongly prime ring and let β denote the strongly prime radical in R-mod. Then*

$$\beta(M) = \{m \in M \mid \text{each } fm\text{-system } X \text{ with } m \in X \text{ has } 0 \in X\}$$

holds for each module M. Furthermore $\beta(M)$ is the unique smallest submodule of M with the property that $M/\beta(M)$ is strongly prime or zero.

PROOF. The last sentence follows by the preceding remark. Write

$$B = \{m \in M \mid \text{each } fm\text{-system } X \text{ with } m \in X \text{ has } 0 \in X\}.$$

If $\beta(M) \neq M$ then $M - \beta(m)$ is an *fm*-system which does not contain zero so $B \subseteq \beta(M)$ in this case. This clearly holds if $\beta(M) = M$.

Now suppose $m \notin B$; we must show $m \notin \beta(M)$. There is an *fm*-system X with $m \in X$ and $0 \notin X$. Let $\mathbb{S} = \{K \subseteq M \mid K \text{ a submodule and } K \cap X = \varnothing\}$. Then $0 \in \mathbb{S}$ and, by Zorn's lemma, we may choose a maximal member K of $\mathbb{S}$. Since $m \notin K$ we are finished if we can show that M/K is strongly prime, equivalently that $M - K$ is an *fm*-system. Given $m_1 \in M - K$ then $Rm_1 + K$ meets X by the maximality of K, say $rm_1 + k = x \in X$. Since X is an *fm*-system, choose a finite set $F = \{r_1, \ldots, r_t\} \subseteq R$ such that $sFx \cap X \neq \varnothing$ for each $0 \neq s \in R$. If $sr_ix \in X$ for such an s, then $sr_irm_1 + sr_ik \in X$. But $K \cap X = \varnothing$ and $sr_ik \in K$ so it follows that $sr_irm_1 \notin K$. Thus

$$sr_irm_1 \in s(Fr)m_1 \cap (M - K)$$

and so $M - K$ is an *fm*-system as required. □

Note that this argument yields slightly more. If X_0 is any *fm*-system with $0 \notin X_0$ then Zorn's lemma produces a maximal *fm*-system $X \supseteq X_0$ with $0 \notin X$. If we now choose K as in the proof of Theorem 2 then $X \subseteq M - K$ (since $X \cap K = \varnothing$) and hence $X = M - K$ by the maximality of X. Thus

COROLLARY. *If X is a maximal fm-system such that $0 \notin X$ in a module M then $K = M - X$ is a submodule with M/K strongly prime. In particular, a module M contains an fm-system X with $0 \notin X$ if and only if M has a strongly prime image.*

We now turn to a characterization of the strongly prime radical which is analogous to the lower radical construction of the prime radical of a ring. Given a module M, inductively define an ascending chain of submodules M_λ, λ an ordinal, as follows:

(1) $M_0 = 0$;

(2) if λ is a limit ordinal, define $M_\lambda = \bigcup_{\mu<\lambda} M_\mu$;

(3) if $\lambda = \mu + 1$, define

$$M_\lambda = M_{\mu+1} = \left\{ m \in M \;\middle|\; \begin{array}{l} \text{given a finite nonempty subset } F \subseteq R, \\ \text{there exists } 0 \neq r \in R \text{ such that } rFm \subseteq M_\mu \end{array} \right\}.$$

Clearly $M_\lambda \subseteq M_{\lambda+1}$ so these M_λ are an ascending chain of submodules. If γ is the least ordinal for which $M_\gamma = M_{\gamma+1}$ write $M_\gamma = L(M)$.

THEOREM 3. *If R is left strongly prime and β is the strongly prime radical in R-mod then $\beta(M) = L(M)$ holds for every $M \in R$-mod.*

PROOF. Let $L(M) = M_\gamma = M_{\gamma+1}$. We show first that $\beta(M) \subseteq M_\gamma$. If $M_\gamma = M$ this is clear. Otherwise it suffices to show M/M_γ is strongly prime. If $m \in M - M_\gamma$ then $m \notin M_{\gamma+1}$ so there exists a finite set $F = \{r_1, \ldots, r_k\} \subseteq R$ such that $rFm \not\subseteq M$ for every $0 \neq r \in R$. Thus $rFm \cap (M - M_\gamma) \neq \varnothing$ for all $0 \neq r \in R$ so $M - M_\gamma$ is an *fm*-system as required.

To prove $M_\gamma \subseteq \beta(M)$ we prove inductively that $M_\lambda \subseteq \beta(M)$ holds for every ordinal λ. The only case where proof is required is when $\lambda = \mu + 1$ for some ordinal μ. Assume $M_\mu \subseteq \beta(M)$ and suppose $m \in M_\lambda - \beta(M)$. Then, since $M - \beta(M)$ is an *fm*-system, there exists a finite set $F \subseteq R$ with $rFm \cap [M - \beta(M)] \neq \varnothing$ for all $0 \neq r \in R$. But $m \in M_\lambda = M_{\mu+1}$ means there exists $0 \neq r_0 \in R$ such that $r_0Fm \subseteq M_\mu$. This contradiction shows that $M_\lambda \subseteq \beta(M)$ and so completes the induction. □

One important class of strongly prime rings is the class of domains. We now relate $\beta(M)$ for $M \in R$-mod to the set of torsion elements $\tau(M)$ where R is a domain. Recall that $\tau(M) = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$.

PROPOSITION 5. *If R is a domain then $\beta(M) \subseteq \tau(M)$ for all $M \in R$-mod.*

PROOF. We use Theorem 3 and show inductively that $M_\lambda \subseteq \tau(M)$ for every ordinal λ. Again we need only discuss the case when $\lambda = \mu + 1$ for some ordinal μ and $M_\mu \subseteq \tau(M)$. Let $m \in M_\lambda$. Then the definition of $M_{\mu+1}$ (with $\{1\} = F$) shows that there exists $0 \neq r \in R$ with $rm \in M_\mu$. Thus $srm = 0$ for some $0 \neq s \in R$ and, since R is a domain, this shows that $m \in \tau(M)$. □

In the case of left Ore domains, Levy [2] has shown that, for each $M \in R$-mod, $\tau(M)$ is a submodule of M. In this case it is easy to verify that τ is a left exact radical on R-mod and it is clear that $\tau(R) = 0$. Hence, by Theorem 1, $\beta \geqslant \tau$. With Proposition 5 this gives:

PROPOSITION 6. *If R is a left Ore domain, then $\tau(M) = \beta(M)$ for all $M \in R$-mod, that is $\tau = \beta$ on R-mod.*

3. The faithful prime radical. The preceding work can be repeated to deal with the radical determined by the class $\mathfrak{M}_0$ of faithful prime modules in R-mod (so we assume R is a prime ring). Then Proposition 2 is valid for $\mathfrak{M}_0$ and yields a left exact radical

$$\beta_0(M) = \{K \mid K \subseteq M, M/K \text{ is faithful and prime}\}$$

when we set $\beta_0(M) = M$ if M has no faithful, prime images. We call $\beta_0(M)$ the *faithful prime radical* of M. Clearly $\beta_0 \leqslant \beta$ over a strongly prime ring.

Define an *m-system* in a module M to be a nonempty subset X of M such that, for each $x \in X$ and $0 \neq r \in R$, $rRx \cap X \neq \varnothing$. Then M/N is a faithful prime module if and only if $M - N$ is an m-system. Furthermore Theorem 2 has its analog for prime rings R obtained by replacing β, "strongly prime" and "*fm*-system" by β_0, "faithful prime" and "m-system" throughout. The proof is analogous to the above and is omitted.

We also have a lower radical construction of $\beta_0(M)$. A sequence M_λ, λ an ordinal, of submodules of a module M is defined as before except that, when $\lambda = \mu + 1$, we define $M_{\mu+1} = \{m \in M |$ there is an ideal $I \neq 0$ of R with $Im \subseteq M_\mu\}$. Again we find that the terminal module in this ascending chain is $\beta_0(M)$.

Finally, let F be a field with a monomorphism $\alpha: F \to F$ which is not onto and let $R = F[x, \alpha]$ be the skew polynomial ring with coefficients written on the left. Then R is a left Ore domain which is left primitive. In fact, if $b \in F - F\alpha$, then $M = R/R(x + b)$ is a faithful irreducible module which is torsion. Hence $\beta_0(M) = 0$ while $\tau(M) = \beta(M) = M$ and so $\beta_0 < \beta$.

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