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THE UNIQUENESS CLASS FOR THE CAUCHY PROBLEM FOR PSEUDOPARABOLIC EQUATIONS

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ABSTRACT. It is shown that the class of functions satisfying $|u(x, t)| < Me^{\alpha|x|}$ forms a uniqueness class for the Cauchy problem for pseudoparabolic equations. The surprising fact is that, unlike the case of parabolic equations, the constant α is not arbitrary but depends on the coefficients of the equation.

Introduction. This paper is concerned with determining the uniqueness class for the Cauchy problem for the pseudoparabolic equation

$$Lu - Mu_t - c(x)u_t + Mu = 0. \tag{1.1}$$

Here M is an elliptic partial differential operator of second order and $c(x)$ is a positive coefficient. All coefficients are assumed to be bounded.

Since solutions of (1.1) are closely related to solutions of the associated parabolic equation

$$Pu = c(x)u_t - Mu = 0 \tag{1.2}$$

(cf. [1], [3]), one would expect a similarity in the uniqueness class for the Cauchy problem, that is a solution that takes prescribed values on the axis $t = 0$.

For the parabolic equation if $|u(x, t)| < Ce^{\alpha x^2}$ for arbitrary fixed α, then there is a unique solution for the Cauchy problem. For equation (1.1) we shall see that the uniqueness class consists of functions of first order growth, that is, $|u(x, t)| < ce^{\alpha|x|}$. The surprising fact is that α in this case is no longer arbitrary but depends on the operator L. More specifically it depends on the lower bound for the coefficient $c(x)$ and the modulus of ellipticity of the operator M. We shall illustrate this with an example, deferring the statement of the main theorem until the next section.

EXAMPLE. There is a nontrivial solution to the Cauchy problem,

$$u_{xxt} - u_t + u_{xx} = 0, \tag{1.3}$$

$$u(x, 0) = 0, \tag{1.4}$$

that satisfies the estimate $|u(x, t)| < e^{\sqrt{2}\,|x|}$.

We let

$$u(x, t) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} a_n(t) \tag{1.5}$$

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where

$$a_n'(t) - a_{n-1}'(t) = a_n(t), \qquad n = 0, 1, 2, \dots,$$
$$a_n(0) = 0, \qquad n = 0, 1, 2, \dots. \tag{1.6}$$

Under these conditions it is easily verified that $u(x, t)$ is a solution to (1.3) and (1.4). We choose $a_0(t)$ to satisfy $0 < a_0(t) < 1$.

Equations (1.6) become

$$a_n(t) = a_{n-1}(t) - \int_0^t e^{-(t-\tau)} a_{n-1}(\tau)\, d\tau,$$

that is

$$a_n(t) = (I - T)^n a_0(t)$$

where

$$Tf = \int_0^t e^{-(t-\tau)} f(\tau)\, d\tau.$$

Thus

$$a_n(t) = \sum_{r=0}^{n} \binom{n}{r} (-1)^r T^r a_0$$

where

$$T^r a_0 = \frac{1}{(r-1)!} \int_0^t (t-\tau)^{r-1} e^{-(t-\tau)} a_0(\tau)\, d\tau.$$

Thus

$$|T^r a_0(t)| < 1, \qquad r = 1, 2, \dots, \qquad t > 0.$$

The last estimate follows by taking the maximum of the integrand and using Stirling's formula. Thus $a_n(t)$ satisfies

$$a_n(t) < 2^n, \qquad n = 0, 1, \dots, \qquad t > 0,$$

and equation (1.5) yields the estimate

$$|u(x, t)| < e^{\sqrt{2}\,|x|}.$$

We first introduce some notation. We denote by D_R the ball $\{x \in \mathbf{R}^n: |x| < R\}$ and by $C^{k+\alpha}(D_R)$ the Banach space consisting of those functions whose derivatives of order k are Hölder continuous in D_R with exponent α, $0 < \alpha < 1$. It is assumed that this space is equipped with its usual norm. By $C_0^{k+\alpha}(D_R)$ we mean the closed subset of $C^{k+\alpha}(D_R)$ consisting of those functions that vanish on the sphere $|x| = R$.

We shall consider solutions $u(x, t)$ of the equation

$$Lu = (M - cI)u_t + Mu = 0 \tag{2.1}$$

for $x \in \mathbf{R}^n$, $t > 0$. Here I is the identity map and M is an elliptic partial differential operator of the form

$$M = -\sum_{i,j=1}^{n} m_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} m_i(x)\frac{\partial}{\partial x_i} + m(x) \tag{2.2}$$

such that

(a) the coefficients lie in $C^{\alpha}(D_R)$ for each $R > 0$ and there is a constant K such that $|m_{ij}(x)| < K$, $|m_i(x)| < K$, $0 < m(x) < K$, $x \in \mathbf{R}^n$, $1 < i,j < n$;

(b) there is a constant $m_0 > 0$ such that

$$\sum_{i,j=1}^{n} m_{ij}(x)\xi_i\xi_j > m_0 \sum_{i=1}^{n} \xi_i^2$$

whenever x and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$.

We shall also assume that the coefficient $c(x)$ lies in $C^{\alpha}(D_R)$ for all $R > 0$ and, for some positive constants μ and K,

$$0 < \mu < c(x) < K, \qquad x \in \mathbf{R}^n. \tag{2.3}$$

By a solution to (2.1) we mean a function $u(x, t)$ such that for each $R > 0$: (i) $u(\cdot, t) \in C^{2+\alpha}(D_R)$ for all $t > 0$, (ii) the map from $[0, \infty)$ to $C^{2+\alpha}(D_R)$, $t \to u(\cdot, t)$ is continuously differentiable, (iii) $u(x, t)$ satisfies (2.1) for $x \in \mathbf{R}^n$ and $t > 0$.

Our main result is the following:

THEOREM. *Let $u(x, t)$ be a solution of* (2.1) *with $u(x, 0) = 0$ that satisfies the estimate*

$$|u(x, t)| < Ce^{\alpha|x|}, \qquad x \in \mathbf{R}^n, \quad t > 0,$$

for some constants C and α with $\alpha < \sqrt{\mu/m_0}$. Then $u(x, t) = 0$ for $(x, t) \in \mathbf{R}^n \times [0, \infty)$.

PROOF. Fix $R > 0$, and let $\phi_\lambda(x)$ satisfy

$$\begin{aligned} M\phi_\lambda &= \lambda^2 m_0 \phi_\lambda, \qquad 0 < \lambda < \sqrt{\mu/m_0}, \quad |x| < R, \\ \phi_\lambda &= e^{\lambda R}, \qquad |x| = R. \end{aligned} \tag{2.4}$$

Then an application of the maximum principle for elliptic operators shows that $\phi_\lambda(x) > 0$ for $x \in D_R$. The presence of the constant m_0 in (2.4) means that we have essentially divided equation (2.1) by m_0 in order to "normalize" the operator M.

If we put

$$\psi(x, t) = Ce^{(\alpha-\lambda)R} e^{\lambda^2 m_0 t(\mu - \lambda^2 m_0)^{-1}} \phi_\lambda(x), \tag{2.5}$$

then

$$v = \psi - u \tag{2.6}$$

satisfies

$$\begin{aligned} Lv &= f(x, t), \qquad x \in D_R, \quad t > 0, \\ v(x, 0) &= h(x), \qquad x \in D_R, \\ v(x, t) &= g(t), \qquad |x| = R, \quad t > 0, \end{aligned} \tag{2.7}$$

where

$$f(x, t) = Ce^{(\alpha-\lambda)R}e^{\lambda^2 m_0 t(\mu-\lambda^2 m_0)^{-1}}\left\{\frac{\lambda^2 m_0(\mu - c(x))}{\mu - \lambda^2 m_0}\right\}\phi_\lambda(x) \leqslant 0, \quad (2.8)$$

$$h(x) = Ce^{(\alpha-\lambda)R}\phi_\lambda(x) \geqslant 0, \quad (2.9)$$

$$g(t) \geqslant Ce^{\alpha R}e^{\lambda^2 m_0 t(\mu-\lambda^2 m_0)^{-1}} - Ce^{\alpha R} \geqslant 0. \quad (2.10)$$

We wish to show that $v(x, t) \geqslant 0$ for $x \in D_R$, $t \geqslant 0$. Unlike the case of a parabolic equation we are unable to achieve this by appealing directly to a maximum principle (cf. [2]). We shall instead convert the initial boundary value problem (2.7) into an integral equation with positive kernel and free term.

We define $G(x, t) \in C^{2+\alpha}(D_R)$ by

$$(M - c(x)I)G = 0, \qquad |x| < R, \quad t > 0,$$
$$G(x, t) = e^t g(t), \qquad |x| = R, \quad t > 0. \quad (2.11)$$

Again by the maximum principle for elliptic operators we have that $G \geqslant 0$ in $D_R \times [0,\infty)$. As a function of t, $G(\cdot,t)$ is continuously differentiable for $t \geqslant 0$. If we make the transformation

$$w(x, t) = e^t v(x, t) - G(x, t), \quad (2.12)$$

then $w(x, t) \in C_0^{2+\alpha}(D_R)$, $w(\cdot,t)$ is continuously differentiable in t and satisfies

$$(M - c(x)I)w_t + c(x)w = e^t f - G, \qquad |x| < R, \quad t > 0,$$
$$w(x, 0) = h(x) - G(x, 0), \qquad |x| \leqslant R,$$
$$w(x, t) = 0, \qquad |x| = R, \quad t \geqslant 0. \quad (2.13)$$

If we rewrite this in the space $C(D_R)$ we obtain

$$w_t - Aw = B(e^t f - G),$$
$$w(0) = h - G(\cdot,0), \quad (2.14)$$

where A and B denote the operators from $C^\alpha(D_R)$ to $C_0^{2+\alpha}(D_R)$ defined by

$$Au = -(M - c(x)I)^{-1}c(x)u, \quad (2.15)$$

$$Bu = -(M - c(x)I)^{-1}u. \quad (2.16)$$

The maximum principle shows that $u \geqslant 0$ pointwise in D_R implies that $Au \geqslant 0$ and $Bu \geqslant 0$ pointwise.

If we integrate (2.14) with respect to t from 0 to t we obtain

$$w(t) = \int_0^t Aw\, d\tau + H(t) \quad (2.17)$$

where

$$H(t) = \int_0^t B(G - e^\tau f)d\tau + h - G(,0). \quad (2.18)$$

We note that the integrand in (2.18) is nonnegative and hence $H(t)$ will be nonnegative provided $h(x) - G(x, 0) \geqslant 0$ in D_R. However the function $\theta(x) = h(x) - G(x, 0) \in C_0^{2+\alpha}(D_R)$ satisfies

$$(M - c(x)I)\theta(x) = Ce^{(\alpha-\lambda)R}[\lambda^2 m_0 - c(x)]\phi_\lambda(x) \leqslant 0 \qquad (2.19)$$

since $\lambda^2 m_0 < \mu \leqslant c(x)$ for $x \in D_R$ and $\phi_\lambda(x) \geqslant 0$ in D_R. The maximum principle now implies that $\theta(x) \geqslant 0$.

Picard iteration applied to (2.17) shows that $w(x, t) \geqslant 0$ for $x \in D_R$, $t \geqslant 0$. The positivity of G and (2.12) yields the desired conclusion that $v(x, t) \geqslant 0$ in $D_R \times [0,\infty)$.

Thus for each $R > 0$

$$u(x, t) \leqslant \psi(x, t) = Ce^{(\alpha-\lambda)R}e^{\lambda^2 m_0 t(\mu-\lambda^2 m_0)^{-1}}\phi_\lambda(x). \qquad (2.20)$$

For any $\alpha < \sqrt{\mu/m_0}$ we may choose a λ such that $\alpha < \lambda < \sqrt{\mu/m_0}$ and let $R \to \infty$ in (2.20) to obtain $u(x, t) \leqslant 0$ in $D_R \times [0,\infty)$. Applying the above analysis to $-u$ yields the conclusion of the theorem.

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