PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 2, September 1979

QUOTIENTS OF c_0 ARE ALMOST ISOMETRIC TO SUBSPACES OF c_0

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ABSTRACT. It is shown that for every $\varepsilon > 0$ and quotient space X of c_0 there is a subspace Y of c_0 such that the Banach-Mazur distance $d(X, Y)$ is less than $1 + \varepsilon$. This improves a result of Johnson and Zippin.

0. Introduction. Johnson and Zippin [2] have shown that a quotient space of c_0 is isomorphic to a subspace of c_0. Here we strengthen this result by showing that a quotient of c_0 is almost isometric to a subspace of c_0, i.e., if X is a quotient of c_0 and $\varepsilon > 0$, then there is an isomorphism T from X into c_0 such that $\|T\|\,\|T^{-1}\| < 1 + \varepsilon$.

We will use standard Banach space notation as may be found in the book of Lindenstrauss and Tzafriri [3].

We wish to thank Y. Benyamini for suggesting several simplifications of our arguments.

1. Proof of the result.

THEOREM. *Let X be a quotient of c_0; then, for every $\varepsilon > 0$, there is a subspace Y of c_0 such that*

$$d(X, Y) = \inf\{\|T\|\,\|T^{-1}\|: T: X \to Y \text{ is an isomorphism}\} < 1 + \varepsilon.$$

PROOF. We will construct a sequence $\{x_i': i \in \mathbf{N}\}$ in B_{X^*} such that $w^* \lim x_i' = 0$ and

$$\sup\{|x_i'(z)|: i \in \mathbf{N}\} > (1 + \varepsilon)^{-1}\|z\|$$

for all $z \in X$. Once this is accomplished, the evaluation map $E: X \to c_0$, defined by $(Ez)(i) = x_i'(z)$, $i \in \mathbf{N}$, gives the required isomorphism.

First we will choose a sequence $\{x_i: i \in \mathbf{N}\}$ such that

$$\sup\{|x_i(z)|: i \in \mathbf{N}\} > (1 - \varepsilon/24)\|z\|,$$

for all $z \in X$, and then show that a slight modification of this sequence converges w^* to zero. Let P_j be the natural projection onto the span of the first j unit vectors of the usual basis of l_1 and let $\tau = \varepsilon/16$ and $\delta = 2\tau^2/3$. Suppose we have chosen integers $n(1), n(2), \ldots, n(i-1)$ and unit vectors $x_1, x_2, \ldots, x_{n(i-1)}$ in X^* such that

Received by the editors September 12, 1978.

AMS (MOS) subject classifications (1970). Primary 46A45; Secondary 46E15.

Key words and phrases. Quotient space, almost isometric.

[1]Supported in part by NSF-MCS 7610613.

$$\|P_j x_l\| \geq 1 - \delta/4 \qquad \text{if } l \leq n(j)$$

and such that if $x \in B_{X^*}$ and $\|P_j x\| \geq 1 - \delta/4$ either

(1) there is an element $z = \sum_{i=1}^{n(j-1)} \lambda_i x_i$, $\lambda_i \geq 0$, such that $\sum_{i=1}^{n(j-1)} \lambda_i \geq \tau$ and

$$\|x - z\| + \sum_{i=1}^{n(j-1)} \lambda_i < (1 + \delta)\|x\|$$

or

(2) there is an element x_l, $n(j-1) < l \leq n(j)$, such that for some λ, $1 \geq \lambda \geq 1 - \delta/4$,

$$\|x - \lambda x_l\| + \lambda \leq (1 + \delta)\|x\|.$$

We choose $n(i)$ and unit vectors $x_{n(i-1)+1}, \ldots, x_{n(i)}$ so that $\{\lambda_l x_l : n(i-1) < l \leq n(i)\}$ is a finite $5\delta/8$ net in

$$A_i = \{x : x \in B_{X^*}, \|P_i x\| \geq 1 - \delta/4, \text{ and (1) is not satisfied}\},$$

for some sequence of nonnegative real numbers $\{\lambda_l : n(i-1) < l \leq n(i)\}$.

Now if $x \in B_{X^*}$, $\|P_i x\| \geq 1 - \delta/4$, and (1) is not satisfied then there is an index l, $n(i-1) < l \leq n(i)$, such that $\|x - \lambda_l x_l\| < 5\delta/8$. Thus

$$\|x - \lambda_l x_l\| + \lambda_l < 5\delta/8 + 1 \leq (1 + \delta)\|x\| \qquad \text{if } \delta \leq \tfrac{1}{2}.$$

The above procedure inductively defines our sequence $\{x_i : i \in \mathbf{N}\}$. Our next task is to verify that

$$\sup\{|x_i(z)| : i \in \mathbf{N}\} \geq (1 - \varepsilon/24)\|z\|$$

for all $z \in X$. This is equivalent to showing that

$$\overline{\mathrm{co}}\,(\{\pm x_i : i \in \mathbf{N}\} \cup \{0\}) \supset (1 - \varepsilon/24) B_{X^*}.$$

It follows from (1) and (2) that if $x \in B_{X^*}$ and $\|P_i x\| \geq 1 - \delta/4$ then there is an element $z = \sum_{i=1}^{n(i)} \lambda_i x_i$, $\lambda_i \geq 0$, such that

$$\|x - z\| + \sum_{i=1}^{n(i)} \lambda_i < (1 + \delta)\|x\| \quad \text{and} \quad \sum_{i=1}^{n(i)} \lambda_i \geq \tau.$$

Thus

$$\|x - z\| < (1 + \delta)\|x\| - \tau \leq (1 + \delta - \tau)\|x\|.$$

Because $\tau > \delta$, we can construct a series $\sum_{i=1}^{\infty} \beta_i x_i$, which converges to x in norm by imitating the standard proof of the open mapping theorem (e.g., [1, p. 56]). Consequently there exists a constant K such that

$$|x| = \inf\left\{ \sum_{i=1}^{\infty} \beta_i : \sum_{i=1}^{\infty} \beta_i x_i = x, \beta_i \geq 0 \right\} \leq K\|x\|,$$

for all $x \in X^*$.

We claim that we can choose $K \leq \tau(\tau - \delta)^{-1} = 24/(24 - \varepsilon)$ and that

(3)

$$|x| < \tau(\tau - \delta)^{-1}\|x\|.$$

Suppose $\rho > 0$, K is minimal, $|x| > K - \rho$, and $\|x\| = 1$. There is an integer j such that $\|P_j x\| > 1 - \delta/4$ and thus by (1) or (2) there is an element $z = \sum_{i=1}^{n(j)} \gamma_i x_i$, $\gamma_i > 0$, such that

$$\|x - z\| + \sum_{i=1}^{n(j)} \gamma_i < 1 + \delta \quad \text{and} \quad \sum_{i=1}^{n(j)} \gamma_i > \tau.$$

Consequently

$$\begin{aligned} |x| + (K-1)\tau &< |x - z| + |z| + (K-1)\sum_{i=1}^{n(j)} \gamma_i \\ &< K\|x - z\| + \sum_{i=1}^{n(j)} \gamma_i < K(1+\delta). \end{aligned}$$

Hence $K - \rho + (K-1)\tau < K(1+\delta)$, for all $\rho > 0$, and $K < \tau(\tau - \delta)^{-1}$, as claimed. If $K < \tau(\tau - \delta)^{-1}$, (3) is obvious. If $K = \tau(\tau - \delta)^{-1}$, replacing K by $\tau(\tau - \delta)^{-1}$ above yields

$$|x| + \left(\tau(\tau - \delta)^{-1} - 1\right)\tau < \tau(\tau - \delta)^{-1}(1 + \delta)$$

or $|x| < \tau(\tau - \delta)^{-1}$, proving (3).

Our final task is to show that there are elements $\{w_i : i \in \mathbf{N}\} \subset (\varepsilon/16)B_{X^*}$ such that $w^* \lim(x_i - w_i) = 0$. Then we can let $x_i' = x_i - w_i$ and

$$\begin{aligned} \sup\{|x_i'(z)| : i \in \mathbf{N}\} &> \sup\{|x_i(z)| : i \in \mathbf{N}\} - (\varepsilon/16)\|z\| \\ &> (1 - \varepsilon/24)\|z\| - (\varepsilon/16)\|z\| \\ &= (1 - 5\varepsilon/48)\|z\| > (1+\varepsilon)^{-1}\|z\| \end{aligned}$$

(if $\varepsilon < 1$).

Let x be a w^* cluster point of $\{x_i : i \in \mathbf{N}\}$ and for notational convenience assume that $w^* \lim x_i = x$. We claim that $\|x\| < \varepsilon/16$. From (3) it follows that there is a sequence $\{\lambda_i : i \in \mathbf{N}\}$ of nonnegative real numbers such that

$$x = \sum_{i=1}^{\infty} \lambda_i x_i \quad \text{and} \quad \|x\| < \sum_{i=1}^{\infty} \lambda_i < \tau(\tau - \delta)^{-1}\|x\|.$$

Suppose $\|x\| > \varepsilon/16$. Let

$$a = \left(\sum_{i=1}^{\infty} \lambda_i\right)^{-1} \varepsilon/16 < 1,$$

and consider

$$\|x_j - ax\| + a\sum_{i=1}^{\infty} \lambda_i, \qquad j = 1, 2, \ldots.$$

For any $\rho > 0$ there is a sufficiently large integer j_0 such that for all $j > j_0$,

$$\|x_j - ax\| < \|x_j\| - a\|x\| + \rho.$$

Thus, if $\rho < \delta - (1 - \|x\|(\Sigma\lambda_i)^{-1})\varepsilon/16$,

$$\|x_j - ax\| + a\sum_{i=1}^{\infty}\lambda_i < \|x_j\| - a\|x\| + \rho + a\sum_{i=1}^{\infty}\lambda_i$$

$$< \|x_j\| - \left(\|x\|\left(\sum_{i=1}^{\infty}\lambda_i\right)^{-1} - 1\right)\frac{\varepsilon}{16} + \rho$$

$$< \|x_j\| + \delta = (1 + \delta)\|x_j\|,$$

for all large j. Clearly we can replace x by some approximate $\Sigma_{i=1}^{i_0}\lambda_i x_i$ to get that, for sufficiently large j,

$$\left\|x_j - a'\sum_{i=1}^{i_0}\lambda_i x_i\right\| + a'\sum_{i=1}^{i_0}\lambda_i < (1 + \delta)\|x_j\|$$

where

$$a' = \left(\sum_{i=1}^{i_0}\lambda_i\right)^{-1}\frac{\varepsilon}{16} = \left(\sum_{i=1}^{i_0}\lambda_i\right)^{-1}\tau.$$

This contradicts the fact that $x_j \in A_m$ if $i_0 < n(m-1) < j \leq n(m)$, and therefore $\|x\| \leq \varepsilon/16$.

For each i let w_i be a w^* cluster point of $\{x_i: i \in \mathbf{N}\}$ such that $\bar{d}(x_i, w_i) < i^{-1} + \inf\{\bar{d}(x_i, w)$: w is a w^* cluster point of $\{x_i: i \in \mathbf{N}\}\}$ where $\bar{d}(\ ,\ )$ is a translation invariant metric compatible with the w^* topology on B_{X^*}. Clearly $w^* \lim(x_i - w_i) = 0$, completing the proof.

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