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THE SOLVABILITY OF OPERATOR EQUATIONS WITH ASYMPTOTIC QUASIBOUNDED NONLINEARITIES[1]

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ABSTRACT. We study the solvability of operator equations involving quasibounded and asymptotically quasibounded nonlinear perturbations of linear Fredholm operators.

1. Let X and Y be Banach spaces, L: $X \to Y$ a linear Fredholm map of nonnegative index and N: $X \to Y$ a compact map. The operator equation of the form

$$Tx = Ax + Nx = f \tag{1}$$

has been extensively studied by many authors in recent years. Under various growth conditions on N, the surjectivity of T has been proven in a number of papers (see **[4]**, **[5]**, **[7]** and the references therein).

Alternatively, beginning with a paper of Landesman and Lazer **[6]**, much work has been done on the solvability of equation (1) for a certain range of values of Pf, where P is the projection of Y on the cokernel of A. Using the stable homotopy arguments, Nirenberg **[9]**, **[10]**, Berger **[1]**, Mawhin **[8]**, Podolak **[11]**, Borisovich, Zvyagin and Sapronov **[2]** and others have studied equation (1). The alternative method has also been used to study equation (1) (with noncompact N too) in a series of papers by Cesari and his coworkers, Fučik, Kučera and Nečas **[5]**, and many others (cf. the survey paper by Cesari **[3]** and the monograph by Berger **[1]** for contributions of other authors). In all these papers (except in **[2]**, **[7]**, **[11]**) N is assumed to have less than linear or linear growth.

In **[2]** and **[11]** the authors have studied equation (1) under the assumption that N is asymptotically linear or asymptotically Lipschitz (i.e., B in Definition 1 below is a Lipschitz map), respectively. In a series of papers Mawhin (cf. **[7]**, **[8]**) has studied equation (1) with $f \in R(A)$ involving certain quasibounded maps N using his coincidence degree.

In this paper we study the surjectivity of T with N either quasibounded or asymptotically quasibounded as defined below. Moreover, in case when the index of A, $i(A)$, is zero we provide a new growth condition on $PN|_{\ker A}$ that insures the solvability of equation (1) with these types of nonlinearities N. In the proofs of our main results we use a special case of the degree theory for

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compact perturbations of nonlinear C^1-Fredholm maps as developed in [2] or, equivalently, the stable homotopy arguments since for our map T this degree can be defined in terms of elements of the stable homotopy group $\pi_{n+m}(S^m)$ (see [1], [2], [9]).

2. Set $X_1 = \ker A$ and $Y_2 = A(X)$. Since A is Fredholm, $\dim X_1 = n < \infty$ and Y_2 is closed we have the following direct sum decompositions: $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ with $\dim Y_1 = m < \infty$ and $\operatorname{ind}(A) = n - m \geq 0$. Define a new norm on X by

$$\|x\|_1 = \max\{\|x_1\|, \|x_2\|\},$$

where $x = x_1 + x_2$ with $x_i \in X_i$, $i = 1,2$. Let P: $Y \to Y_1$ be a linear continuous projection onto Y_1, H be the inverse of the linear homeomorphism $A|_{X_2}$: $X_2 \to Y_2$ and $\alpha = \|H\|$.

THEOREM 1. *Suppose that for a given f in Y the following conditions hold*:

(1) *There exist constants $M_f > 0$ and $N_f > 0$ such that $PN(x_1 + x_2) - tf_1 \neq 0$ for $\|x_2\| \leq r$, $r > N_f$, $\|x_1\| \geq rM_f$ and $t \in [0, 1]$;*

(2) *$M = H(I - P)N$ is quasibounded, i.e.,*

$$|M| = \limsup_{\|x\|_1 \to \infty} \frac{\|Mx\|}{\|x\|_1} < \infty$$

and $|M| \max\{1, M_f\} < 1$;

(3) *the stable homotopy class η_ρ of $PN|S_\rho^{n-1}$: $S_\rho^{n-1} \to Y_1 \setminus \{0\}$, $\rho \geq rM_f$, is nontrivial, where $S_\rho^{n-1} \subset X_1$ is a sphere of radius ρ.*

Then equation (1) *is solvable for this f.*

PROOF. Let $\varepsilon > 0$ be small. By (2) there exists $R \geq N_f$ such that

$$\|Mx\| = \|H(I - P)Nx\| \leq (|M| + \varepsilon)\|x\|_1$$

for all $\|x\|_1 \geq R$. Moreover, there exists an $r \geq R$ such that $Ax + t(I - P)Nx - tf_2 \neq 0$ for all $x = x_1 + x_2$ with $\|x_1\| < rM_f$ and $\|x_2\| = r$ and $t \in [0, 1]$. If not, then for each $r > R$ there exist $t \in [0, 1]$ and x with $\|x_1\| < rM_f$ and $\|x_2\| = r$ such that $Ax_2 + t(I - P)Nx - tf_2 = 0$, and therefore

$$\|x_2\| \leq \|H(I - P)Nx\| + \alpha\|f_2\| \leq (|M| + \varepsilon)\|x\|_1 + \alpha\|f_2\|,$$

or

$$1 \leq \frac{1}{r}(|M| + \varepsilon)\|x\|_1 + \frac{\alpha}{r}\|f_2\| \leq (|M| + \varepsilon)\max\{1, M_f\} + \frac{\alpha}{r}\|f_2\|.$$

Passing to the limit as $r \to \infty$, we obtain $1 \leq (|M| + \varepsilon)\max\{1, M_f\}$ which is in contradiction with condition (2) for ε small enough. Hence, an r with the above property exists.

Next, we define $\overline{D} = \{x = x_1 + x_2 \in X \mid \|x_1\| \leq rM_f,\ \|x_2\| \leq r\}$ with r chosen as above, and define the homotopy H: $[0, 1] \times \overline{D} \to Y$ by

$$H(t, x) = (Ax + t(I - P)Nx - tf_2, PN(x_1 + tx_2) - tf_1).$$

We claim that $H(t, x) \neq 0$ for $t \in [0, 1]$ and $x \in \partial D$. Indeed, if $x \in \partial D$ is such that $\|x_2\| < r$, then $\|x_1\| = rM_f$ and by (1), $PN(x_1 + tx_2) - tf_1 \neq 0$ for all $t \in [0, 1]$. If $x \in \partial D$ is such that $\|x_1\| < rM_f$, then $\|x_2\| = r$ and $Ax + t(I - P)Nx - tf_2 \neq 0$ for all $t \in [0, 1]$. Thus, by the homotopy theorem in [2],

$$\deg\big(A + N - f, \bar{D}, 0\big) = \deg\big(H_0, \bar{D}, 0\big) = \eta_r,$$

which, by the solvability property of this degree, implies that $Ax + Nx = f$ for some $x \in D$. □

To treat a larger class of nonlinear maps N, we need:

DEFINITION 1. A map $N: X \to Y$ is said to be *asymptotically quasibounded* if there exists a nonzero continuous quasibounded map $B: X \to Y$, i.e.,

$$|B| = \limsup_{\|x\| \to \infty} \frac{\|Bx\|}{\|x\|} < \infty$$

such that

(A) $\lim_{R \to \infty} N(Rx)/R = B(x)$ uniformly on bounded sets in X.

Such maps with B Lipschitz have been studied by Podolak [11].

Theorem 1 admits the following extension:

THEOREM 2. *Suppose that N satisfies condition* (A) *and that B is continuous, satisfies conditions* (1) *and* (3) *of Theorem* 1 *for $f = 0$ and that the following condition holds*:

(2′) *$K = H(I - P)B$ is quasibounded, i.e.*,

$$|K| = \limsup_{\|x\| \to \infty} \frac{\|Kx\|}{\|x\|_1} < \infty$$

and $|K| \max\{1, M_0\} < 1$.

Then equation (1) *is solvable for each f in Y.*

PROOF. Since for each f in Y, $N_f x = Nx - f$ satisfies condition (A) with the same B, it is sufficient to consider the case $f = 0$. Define

$$\bar{D} = \{x = x_1 + x_2 \in X \,|\, \|x_1\| \leq rM_0, \|x_2\| \leq r\},$$

where r is chosen as in Theorem 1 using property (2′) of K. For $R > 0$, define the map $H_R: \bar{D} \to Y$ by

$$H_R(x) = (1/R)(A(Rx) + (I - P)N(Rx), PN(Rx))$$

and the homotopy $H: [0, 1] \times \bar{D} \to Y$ by

$$H(t, x) = (Ax + t(I - P)Bx, PB(x_1 + tx_2)).$$

By our choice of r we know that $H(t, x) \neq 0$ for $t \in [0, 1]$ and $x \in \partial D$. Clearly, if $x \in X$ is a solution of equation (1), then $u = x/R \in D$ is a solution of $H_R(u) = 0$ for R sufficiently large, and conversely. Moreover, $\lim_{R \to \infty} H_R(x) = H(1, x)$ uniformly for $x \in D$ with $\|H(1, x)\| \geq \varepsilon > 0$ for all $x \in \partial D$ since $H(1, \cdot)$ is a proper map. In view of this, it follows that for sufficiently large R, $H_R(x) \neq 0$ on ∂D and

$$F_R(t, x) = H(1, x) + t(H_R(x) - H(1, x)) \neq 0$$

for $t \in [0, 1]$ and $x \in \partial D$. The compactness of N and condition (A) imply that B is compact and consequently

$$F_R(t, x) = Ax + (1 - t)Bx + tN(Rx)/R$$

is an admissible homotopy on $[0, 1] \times \bar{D}$ (cf. (4.2) in [2]). Hence,

$$\deg\big(H_R, \bar{D}, 0\big) = \deg\big(H(1, \cdot), \bar{D}, 0\big) = \deg\big(H(0, \cdot), \bar{D}, 0\big) = \eta_r$$

which implies that the equation $H_R(x) = 0$ is solvable in D. $\square$

REMARK. When N is asymptotically linear, i.e., $N(x) = B(x) + w(x)$, $x \in X$, for some continuous and linear map B: $X \to Y$ with $w(x)/\|x\| \to 0$ as $\|x\| \to \infty$, then N is quasibounded with $|N| = \|B\|$. Hence, Theorem 1 extends Theorem 4.5 in [2], which is, on the other hand, an abstract extension of some results of Nirenberg [9] involving everywhere bounded nonlinearities N. Other extensions of Nirenberg's results to sublinear or quasibounded nonlinearities are given in [1], [4], [5], [7], [8] (cf. [1] for other references).

REMARK. If B in condition (A) is Lipschitz, i.e., $\|Bx - By\| < k\|x - y\|$ for all $x, y \in X$ and some small $k > 0$, then condition (1) in Theorem 2 can be replaced by the following easier to verify condition of Podolak [11]:

(1′) $\|PN(a \cdot x_0)\| > b$ for some positive b and all $a \in R^n$ with $\|a\| = 1$, where $x_0 = \{x_{01}, \ldots, x_{0n}\}$ is a fixed basis for ker A of unit vectors and

$$a \cdot x_0 = a_1x_{01} + \cdots + a_nx_{0n}.$$

In this sense Theorem 2 extends Theorem 1 in [11].

Let us now look at a new condition on $PN|_{X_1}$ which implies that $\deg(PN|_{X_1}, B(0, r), 0) \neq 0$ with $B(0, r) \subset X_1$. Suppose that X and Y are such that there exist a map J: $X_1 \to Y_1^*$ and a continuous and odd map G: $X_1 \to Y_1$ with $Gx \neq 0$ for $x \neq 0$ and $(Gx, Jx) = \|Gx\| \cdot \|Jx\|$ for all $x \in X_1$. This is always so if $Y = X$ or $Y = X^*$. Indeed, if $Y_1 = X_1$, as G and J we can take the identity and the normalized duality map, respectively; while, if $Y_1 = X_1^*$ as G and J we can take the normalized duality map and the identity, respectively. The condition in question is:

(4) $\|PNx\| + (PNx, Jx)/\|Jx\| > 0$ for $x \in \partial B(0, \rho)$, $\rho > rM_f$.

COROLLARY 1. *Let A and N satisfy conditions* (1) *and* (2) *of Theorem* 1. *Then, if condition* (4) *holds for all $\rho > rM_f$ and the index of A is zero, equation* (1) *is solvable.*

PROOF. By Theorem 1 it suffices to show that $\deg(PN, B(0, \rho), 0) \neq 0$, where PN is restricted to $\bar{B}(0, \rho)$. Define the homotopy H: $[0, 1] \times \bar{B}(0, \rho) \to Y_1$ by $H(t, x) = tPNx + (1 - t)Gx$. Then $H(t, x) \neq 0$ for $t \in [0, 1]$ and $x \in \partial B$. If not, then $tPNx + (1 - t)Gx = 0$ for some $t \in [0, 1]$ and $x \in \partial B$. Since $t \neq 0,1$, we have

$$\|PNx\| + \frac{(PNx, Jx)}{\|Jx\|} = \frac{1 - t}{t}\|Gx\| - \frac{1 - t}{t}\,\frac{(Gx, Jx)}{\|Jx\|} = 0$$

in contradiction with condition (4). By the oddness of G we obtain:

$$\deg(PN, B(0, \rho), 0) = \deg(G, B(0, \rho), 0) \neq 0. \quad \square$$

Similarly, using Theorem 2, we obtain:

COROLLARY 2. *Let K be asymptotically quasibounded and B satisfy conditions* (1) *and* (2′) *of Theorem* 2 *with $f = 0$. Then, if* $\operatorname{ind} A = 0$ *and PB satisfies condition* (4) *for $f = 0$, equation* (1) *is solvable for each f in Y.*

Under a somewhat stronger condition than (4), we have:

THEOREM 3. *Let X and Y be Banach spaces with* $\dim X = \dim Y < \infty$ *and let $T: X \to Y$ be continuous and satisfy*

(5) $\|Tx\| + (Tx, Jx)/\|Jx\| \to \infty$ *as* $\|x\| \to \infty$, *where J and G are as above. Then $T(X) = Y$.*

PROOF. Let f in Y be fixed. By condition (5) there exists an $r_f > 0$ such that

$$\|Tx - tf\| > 0 \quad \text{for } \|x\| = r_f, \quad t \in [0, 1]$$

and

$$\|Tx\| + \frac{(Tx, Jx)}{\|Jx\|} > 0 \quad \text{for } \|x\| = r_f.$$

The first inequality implies that

$$\deg(T - f, B(0, r_f), 0) = \deg(T, B(0, r_f), 0),$$

which is nonzero by the second inequality as shown in Corollary 1. Hence, $Tx = f$ is solvable. $\square$

REMARK. Along similar lines one can show that if $T: X \to X$ is continuous and compact (or condensing) and $I - T$ satisfies condition (5), then $(I - T)(X) = X$ (the proof will appear in a forthcoming paper by the author).

Condition (5) for PN clearly holds if PN is coercive on X_1, i.e.,

if $(PNx, Jx)/\|Jx\| \to \infty$ as $\|x\| \to \infty$, $x \in X_1$, or

if $(PNx, Jx) \geqslant -c_1\|Jx\|$ for all $x \in X_1$ and some $c_1 > 0$ and $\|PNx\| \to \infty$ as $\|x\| \to \infty$, $x \in X_1$, and, in particular,

if $\|PNx\| \geqslant c_2\|x\|^k$ for all $x \in X_1$ and some $c_2 > 0$, $k > 0$.

The last condition holds if N is k-homogeneous. Indeed, since $\|PNx\| \neq 0$ for $x \in \partial B(0, r) \subset X_1$,

$$a = \min\{\|PNx\| \mid \|x\| = r\} > 0$$

and $\|PNx\| \geqslant (a/r^k)\|x\|^k$ for all $\|x\| \geqslant r$.

In view of the above discussion, we have the following special case of Theorem 2.1 in **[8]**:

THEOREM 4. *Let $A: D(A) \subset X \to Y$ be a linear Fredholm map of index zero and $N: \overline{D} \subset X \to X$ a continuous compact map, where D is open and bounded. Suppose that*

(i) *$Ax \neq \lambda Nx$ for $x \in D(A) \cap \partial D$ and $\lambda \in (0, 1)$;*

(ii) $PNx \neq 0$ *for each* $x \in \ker A \cap \partial D$;

(iii) *for some isomorphism* $L\colon Y_1 \to X_1$,

$$\|LPNx\| + \frac{(LPNx, Jx)}{\|Jx\|} > 0 \quad \textit{for } x \in \partial D \cap X_1$$

with J *the normalized duality map from* X_1 *to* $2^{X_1^*}$.

Then the equation $Ax - \lambda Nx = 0$ *has at least one solution in* D *for each* $\lambda \in [0, 1]$.

PROOF. It suffices to show (cf. [8]) that deg $(LPN|_{X_1}, D \cap X_1, 0) \neq 0$. But, this follows from condition (iii) as in Corollary 1 since I is odd. □

REMARK. The above results could be proven by using the homotopy

$$H(t, x) = (x_2 + tH(I - P)Nx - tf_2, PN(x_1 + tx_2) - tf_1)$$

instead. Hence, it is sufficient to require that the map $H(I - P)N\colon X \to X$ be compact or condensing. The same observation holds for Theorem 2 with N replaced by B. Moreover, Theorem 2 of Podolak [11] can be shown to be valid for the nonlinearities considered in our Theorem 2.

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