PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 2, September 1979

A COMPLETE BOUNDED COMPLEX SUBMANIFOLD OF $\mathbf{C}^3$

PETER W. JONES

ABSTRACT. We produce an example of a bounded complete complex submanifold of $\mathbf{C}^3$. This is accomplished by using the duality between $H^1(\mathbf{T})$ and BMO($\mathbf{T}$).

The question of whether there exists a complete bounded complex submanifold of $\mathbf{C}^n$ has been an open problem (see [3] for definitions and a discussion of this problem). We present here a method of producing such submanifolds. Suppose that $f_1(z)$ and $f_2(z)$ are two functions which are analytic and bounded in the unit disk Δ of $\mathbf{C}$, and suppose that these two functions have the property

$$\int_\Gamma \{|f_1'(z)| + |f_2'(z)|\}\, d\sigma(z) = \infty \tag{1.1}$$

for all curves $\Gamma \subset \Delta$ which terminate on $\partial\Delta = \mathbf{T}$. (Here σ denotes Euclidean arc length.) Then $z \in \Delta \to (z, f_1(z), f_2(z))$ is an embedded complete bounded complex submanifold of $\mathbf{C}^3$. To construct two bounded analytic functions f_1 and f_2 satisfying (1.1) we use C. Fefferman's theorem [1] that every real valued function $\varphi \in \mathrm{BMO}(\mathbf{T})$ can be represented by $\varphi = u + \tilde{v}$, $u, v \in L^\infty(\mathbf{T})$. Here $\tilde{v}$ denotes the Hilbert transform of v. Consider the harmonic function

$$\varphi(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{r^{10^n}}{n} \cos 10^n\theta.$$

Then $|\nabla\varphi(z)| > 10^n/100n$ if z is in the annulus

$$A_n = \{z: 1 - 11 \cdot 10^{-n-1} < |z| < 1 - 9 \cdot 10^{-n-1}\}.$$

To see this, note that $|\nabla((1/n)r^{10^n} \cos 10^n\theta)|$ is of order of magnitude $10^n/n$ on A_n. The term

$$\left|\nabla\left(\sum_{j=1}^{n-1} \frac{r^{10^j}}{n} \cos 10^j\theta\right)\right|$$

is small on A_n because it is bounded pointwise by $2\sum_{j=1}^{n-1} 10^j/j$. The term

$$\left|\nabla\left(\sum_{j=n+1}^{\infty} \frac{r^{10^j}}{j} \cos 10^j\theta\right)\right|$$

Received by the editors November 30, 1978.

AMS (*MOS*) *subject classifications* (1970). Primary 53B25; Secondary 30A78.

Key words and phrases. Complete bounded complex submanifold, proper mapping, BMO, VMO.

0002-9939/79/0000-0423/$01.50

is small on A_n because it is bounded there by

$$2\sum_{j=n+1}^{\infty} 1/j \cdot 10^j \cdot e^{-(1/2)10^{j-n}}.$$

Now if Γ is a curve in Δ terminating on $\mathbf{T}$,

$$\int_\Gamma |\nabla\varphi(z)|\, d\sigma(z) = \infty,$$

because Γ must cross A_n for all n larger than some integer. It is easy to check by hand that $\varphi(e^{i\theta}) \in \mathrm{BMO}(\mathbf{T})$. (This is clear anyway by Paley's theorem.) By Fefferman's theorem, $\varphi = u + \tilde{v}$ for some $u, v \in L^\infty(\mathbf{T})$. Let $f_1 = e^{u+i\tilde{u}}$ and $f_2 = e^{v+i\tilde{v}}$. Then f_1 and f_2 are in $H^\infty(\Delta)$, and since f_1 and f_2 are bounded from below on Δ,

$$|f_1'(z)| + |f_2'(z)| > c|\nabla\varphi(z)|$$

for some constant c. This means that f_1 and f_2 satisfy property (1.1).

We note that by replacing f_1 by $f_1 + \alpha z$ for a suitable $\alpha \in C$,

$$z \in \Delta \to (f_1(z) + \alpha z, f_2(z))$$

yields a complete bounded immersed curve in C^2. (Just pick α so that $\{z: f_1'(z) = -\alpha\} \cap \{z: f_2'(z) = 0\} = \varnothing$.)

With only a little more work one can produce a *proper* holomorphic mapping φ from Δ to the ball in $\mathbf{C}^4$ such that the image of Δ is a complete complex submanifold. Let $\varphi(e^{i\theta})$ be as before. It is easy to check that $\varphi \in \mathrm{VMO}(\mathbf{T})$ (see [2] for the definition of VMO). By a theorem of Sarason [2], φ can be represented as $\varphi = u + \tilde{v}$, where u and v are continuous on $\mathbf{T}$. Let $f_1 = \varepsilon e^{u+i\tilde{u}}$ and $f_2 = \varepsilon e^{v+i\tilde{v}}$, where ε is chosen so that

$$1 - \varepsilon^2 - |f_1|^2 - |f_2|^2 > \tfrac{1}{2}$$

on $\mathbf{T}$. Let

$$g(e^{i\theta}) = \tfrac{1}{2}\log\left\{1 - \varepsilon^2 - |f_1(e^{i\theta})|^2 - |f_2(e^{i\theta})|^2\right\}.$$

Clearly g is continuous on $\mathbf{T}$. Put $f_3 = e^{g+i\tilde{g}}$. Then $|f_3(z)|$ is continuous and

$$\varepsilon^2 + |f_1(z)|^2 + |f_2(z)|^2 + |f_3(z)|^2 \to 1$$

as $|z| \to 1$. The mapping $\varphi(z) = (\varepsilon z, f_1(z), f_2(z), f_3(z))$ now does the job.

ACKNOWLEDGEMENTS. The author would like to thank Robert Greene for suggesting the first problem treated here, and R. Narasimhan for suggesting the second problem.

REFERENCES

1. C. Fefferman and E. M. Stein, *H^p spaces of several real variables*, Acta Math. **129** (1972), 137–193.

2. D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405.

3. P. Yang, *Curvature of complex submanifolds of* $\mathbf{C}^n$, Proc. Sympos. Pure Math., vol. 30, part 2, Amer. Math. Soc., Providence, R. I., 1977, pp. 135–137.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637