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CHARACTERIZATION OF p-PREDICTORS

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ABSTRACT. Let $(\Omega, \mathcal{Q}, P)$ be a probability space and $1 < p < \infty$. It is shown that each operator T: $L_p(\Omega, \mathcal{Q}, P) \to L_p(\Omega, \mathcal{Q}, P)$ which is homogeneous, constant preserving, positive, quasi-additive and fulfills Dykstra's condition is an p-predictor with respect to a suitable σ-field, i.e. a nearest point projection onto a closed subspace $L_p(\Omega, \mathfrak{B}, P)$, where $\mathfrak{B} \subset \mathcal{Q}$ is a σ-field. None of the conditions for T can be dispensed without compensation.

Let $(\Omega, \mathcal{Q}, P)$ be a probability space and $L_p(\Omega, \mathcal{Q}, P)$ (for $1 < p < \infty$) be the space of all P-equivalence classes of real-valued $\mathcal{Q}$-measurable functions whose absolute pth powers are integrable. Let $\mathfrak{B}$ be a sub-σ-field of $\mathcal{Q}$, then $L_p(\Omega, \mathfrak{B}, P)$–or $L_p(\mathfrak{B})$ for short–is the system of all equivalence classes of $L_p(\Omega, \mathcal{Q}, P)$, containing a $\mathfrak{B}$-measurable function. The operator $P_p^{\mathfrak{B}}$ which assigns to each $f \in L_p(\Omega, \mathcal{Q}, P)$ the unique element in $L_p(\Omega, \mathfrak{B}, P)$ with minimum distance from f is called the p-predictor given $\mathfrak{B}$ (see Ando and Amemiya [2]). $P_p^{\mathfrak{B}}f$ is the unique element of $L_p(\mathfrak{B})$ with

$$\|f - P_p^{\mathfrak{B}}f\|_p \leq \|f - g\|_p$$

for all $g \in L_p(\mathfrak{B})$.

The operator $T = P_p^{\mathfrak{B}}$: $L_p(\Omega, \mathcal{Q}, P) \to L_p(\Omega, \mathcal{Q}, P)$ has the following properties (see [2]):

(1) T is homogeneous, i.e., $T(\alpha f) = \alpha Tf$, for $f \in L_p(\Omega, \mathcal{Q}, P)$ and $\alpha \in \mathbf{R}$;

(2) T is quasi-additive, i.e., $T(f + Tg) = Tf + Tg$, for $f,g \in L_p(\Omega, \mathcal{Q}, P)$;

(3) T is constant-preserving, i.e., $T1 = 1$;

(4) T is positive, i.e., $Tf \geq 0$, for $0 \leq f \in L_p(\Omega, \mathcal{Q}, P)$;

(5) T fulfills Dykstra's condition, i.e. (see [5]), $\|I - T\|_p < 1$, where $\|I - T\|_p$ is defined by

$$\|I - T\|_p = \inf\{c: \|f - Tf\|_p \leq c\|f\|_p\}.$$

In the special case $p = 2$, $P_2^{\mathfrak{B}}f$ is the usual conditional expectation of f given $\mathfrak{B}$ with respect to P. The operators $P_2^{\mathfrak{B}}$ have been characterized by many authors, for instance Bahadur [3], Douglas [4], Moy [7] and Pfanzagl [9]. But as far as we know there does not exist a characterization for $P_p^{\mathfrak{B}}$ if $p \neq 2$. This may be due to the fact that $P_p^{\mathfrak{B}}$ is in general not a linear operator (for a characterization of linearity of $P_p^{\mathfrak{B}}$ see [6]). Now we prove that every operator T: $L_p(\Omega, \mathcal{Q}, P) \to L_p(\Omega, \mathcal{Q}, P)$ which fulfills (1)–(5) is a p-predictor $P_p^{\mathfrak{B}}$ for some suitable sub-σ-field $\mathfrak{B} \subset \mathcal{Q}$.

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THEOREM. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $1 < p < \infty$. Let T: $L_p(\Omega, \mathcal{A}, P) \to L_p(\Omega, \mathcal{A}, P)$ be a homogeneous, quasi-additive, constant preserving, positive operator fulfilling Dykstra's condition. Then there exists a sub-σ-field $\mathcal{B} \subset \mathcal{A}$ such that $T = P_p^{\mathcal{B}}$, i.e., Tf is the nearest point projection of f onto the subspace $L_p(\Omega, \mathcal{A}, P)$.*

PROOF. Let $F = \{f \in L_p(\Omega, \mathcal{A}, P)\colon Tf = f\}$. Since T is homogeneous and quasi-additive, F is a linear space. The same properties imply that T is idempotent and hence $F = \{Tf\colon f \in L_p(\mathcal{A})\}$. Now we show that T is a projection onto F, i.e. we show

$$\|f - Tf\|_p \leq \|f - Tg\|_p \quad \text{for } f,g \in L_p(\mathcal{A}).$$

Using that T is homogeneous, quasi-additive and that $\|I - T\|_p \leq 1$ we obtain

$$\begin{aligned}\|f - Tf\|_p &= \|(f - Tg) - (Tf - Tg)\|_p = \|(f - Tg) - (Tf + T(-g))\|_p \\ &= \|(f - Tg) - T(f + T(-g))\|_p = \|(f - Tg) - T(f - Tg)\|_p \\ &\leq \|f - Tg\|_p.\end{aligned}$$

Now it remains to show that $F = L_p(\mathcal{B})$ for some σ-field $\mathcal{B} \subset \mathcal{A}$. According to Proposition I-1-1 [**8**, p.2], this amounts to verifying that:

(i) $1 \in F$,

(ii) F is closed,

(iii) $f \in F$ implies $f^+ \in F$.

Since T is constant-preserving, (i) holds. To prove (ii), let $f_n \in F$, $n \in \mathbf{N}$, and $\|f_n - f\|_p \underset{n\to\infty}{\to} 0$. Then $Tf_n = f_n$ and the properties of T imply

$$\|f - Tf\|_p = \|f - Tf_n - T(f - Tf_n)\|_p \leq \|f - f_n\|_p \underset{n\to\infty}{\to} 0$$

and hence $Tf = f$, i.e., $f \in F$.

To see (iii), let $f \in F$, i.e. $f = Tf$. We have to prove: $Tf^+ = f^+$. We show at first that $Tf^+ \geq f^+$. Since T is positive it suffices to show $Tf^+ \geq f$. As $f^+ - f \geq 0$, $Tf = f$, and T is positive, homogeneous and quasi-additive we obtain

$$0 \leq T(f^+ - f) = T(f^+ - Tf) = Tf^+ - Tf = Tf^+ - f,$$

i.e., $Tf^+ \geq f$. Let $g = Tf^+ - f^+$. Then $g \geq 0$ and $Tg = 0$. If $g > 0$ on a set of positive P-measure, then $h = P_p^{\{\varnothing,\Omega\}}g > 0$. Since the p-projection on $L_p(\{\varnothing, \Omega\})$ is unique we obtain $\|g - h\|_p < \|g\|_p$. Since $h \in F$ this contradicts $Tg = 0$. Consequently $g = 0$. Hence $Tf^+ = f^+$, i.e., $f^+ \in F$.

Since Bahadur's conditions directly imply our conditions, our result contains the result of Bahadur [3], who proved that a linear idempotent, selfadjoint, positive and constant preserving operator T: $L_2(\mathcal{A}) \to L_2(\mathcal{A})$ is a usual conditional expectation operator. It is easy to see that none of the five properties of T can be dispensed without compensation.

The following example shows, that it is not possible to weaken quasi-additivity to quasi-quasi-additivity (i.e., $T(Tf + Tg) = Tf + Tg$), even if we add three other conditions which were often used by other authors, namely monotony, norm-continuity and Šidák's conditions, i.e.,

$$T(Tf \vee Tg) = Tf \vee Tg$$

(see [**10**, p. 271, Theorem 6]).

EXAMPLE. Let $\Omega = \{1, 2\}$, $\mathcal{Q}$ be the power set of Ω and $P|\mathcal{Q}$ be the probability measure defined by $P(\{1\}) = P(\{2\}) = \frac{1}{2}$. We consider the case $p = 2$. We have

$$L_2(\Omega, \mathcal{Q}, P) = \{\alpha 1_{\{1\}} + \beta 1_{\{2\}}: \alpha,\beta \in \mathbf{R}\}.$$

For each $\alpha,\beta \in \mathbf{R}$ let

$$c(\alpha, \beta) = \operatorname{sign}(\alpha + \beta)\frac{1}{2\sqrt{2}}\frac{(\alpha + \beta)^2}{\sqrt{\alpha^2 + \beta^2}}.$$

Now define T: $L_2(\Omega, \mathcal{Q}, P) \to L_2(\Omega, \mathcal{Q}, P)$ by $T(\alpha 1_{\{1\}} + \beta 1_{\{2\}}) \equiv c(\alpha, \beta)$. Since $c(\alpha, \alpha) = \alpha$, $F = \{f\colon Tf = f\}$ is the set of all constant functions, i.e., $F = L_2(\{\varnothing, \Omega\})$. It is easy to see that T is a homogeneous, constant preserving, positive operator fulfilling Dykstra's condition. The last property immediately follows from $c(\alpha, \beta) < \alpha + \beta$ which is equivalent to $P((f - Tf)^2) < P(f^2)$ for $f = \alpha 1_{\{1\}} + \beta 1_{\{2\}}$ with $(\alpha + \beta) > 0$.

Moreover, T is idempotent, monotone, quasi-quasi-additive, fulfills Šidák's condition and is continuous. But all these properties cannot replace quasi-additivity in the preceding theorem, since T is not the projection onto F. The projection onto F is the usual conditional expectation given $\mathcal{B} = \{\varnothing, \Omega\}$, i.e.,

$$P_2^{\mathcal{B}}(\alpha 1_{\{1\}} + \beta 1_{\{2\}}) = \tfrac{1}{2}(\alpha + \beta).$$

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