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THE COHOMOLOGY OF THE PROJECTIVE n-PLANE[1]

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ABSTRACT. An H-space is a topological space with a continuous multiplication and an identity element. In this paper X has the homotopy type of a countable CW-complex with integral cohomology of finite type and primitively generated k-cohomology, k a field. The projective n-plane of X is denoted $XP(n)$. The main results of this paper are: Theorem 1 which states that $H^*(XP(n)) = N \oplus S$ where N is a truncated polynomial algebra over k and S is a trivial k-ideal, and Theorem 2 which considers the case $k = Z(p)$ and states that $H^*(XP(n)) = \hat{N} \oplus \hat{S}$ where $\hat{N}$ is a truncated polynomial algebra on generators in even dimensions and S is an $A(p)$-subalgebra of $H^*(XP(n))$ so that an $A(p)$-algebra structure can be induced on $\hat{N}$. These theorems extend results by A. Borel, W. Browder, M. Rothenberg, N. E. Steenrod, and E. Thomas.

0. Introduction. An H-space is a topological space with a continuous multiplication and an identity element. In [10] Stasheff defined the projective n-plane of an H-space. In this paper X has the homotopy type of a countable CW-complex with integral cohomology of finite type and primitively generated k-cohomology, k a field. The two main results of this paper pertain to the cohomology of the projective n-plane, $XP(n)$, of an H-space, X. Theorem 1 states that

$$H^*(XP(n)) = N(m) \oplus S$$

where $N(m)$ is a truncated polynomial algebra over k and $S \cup H^*(XP(n)) = 0$. Theorem 2 considers the case $k = Z(p)$ and states that

$$H^*(XP(n)) = \hat{N}(n) \oplus \hat{S}$$

where $\hat{N}$ is a truncated polynomial algebra on generators in even dimensions and $\hat{S}$ is an $A(p)$-subalgebra of $H^*(XP(n))$ so that an $A(p)$-algebra structure can be induced on $\hat{N}$. In a subsequent paper Theorem 2 will be used to study the action of the Steenrod algebra, $A(p)$, on $H^*(XP(n), Z(p))$, and $H^*(X, Z(p))$.

In [2] William Browder and Emery Thomas studied the $Z(2)$-cohomology of $XP(2)$, and in [3] it was pointed out that Borel's methods can be used to obtain the $Z(p)$-cohomology of $XP(\infty)$ when it exists (if and only if the space has an associative multiplication [10]). Steenrod and Rothenberg

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studied $H^*(XP(n))$ for associative H-spaces in [9]. Theorems 1 and 2 extend the results of [2], [3], and [9] by considering $XP(n)$, $n > 2$, for H-spaces other than topological groups.

1. Main results. We begin with some notation. If V is a graded vector space over the field k, then V^o and V^e will denote the subspaces of odd and even dimensional elements respectively. The free commutative algebra generated by V is

$$U(V) = \Lambda(V^o) \otimes k(V^e) \quad \text{if char } k \neq 2$$

and

$$U(V) = k(V) \quad \text{if char } k = 2.$$

The exterior algebra is denoted by Λ and $k(V)$ is the polynomial algebra. Let $U(V/t)$ be the truncated algebra of height t generated by V.

An A_n-structure on X, [10], is a quasi-fibration

$$p_n\colon (E(n), E(n-1), \ldots, X) \to (XP(n-1), XP(n-2), \ldots, *)$$

with fiber X. The space $E(m) = X \circ m \circ X$ is the m-fold join of X, [7], with the usual inclusion into $E(m+1)$, and $XP(m) = c_{p_m}$ the mapping cone of p_m which is p_n restricted to $E(m)$. Let T_m denote the vector space of primitive elements of $H^*(X)$ which are transgressive in the quasi-fibration

$$X \to E(m) \xrightarrow{p_m} XP(m-1).$$

The set $x = \{x_i\}_I$ is a vectorspace basis for T_m. Let $y_i \in H^*(XP(m-1))$ be a transgression of x_i and $\mathfrak{Y}_m = \{y_i\}_I$. The mapping cone of p_m is $XP(m)$ and there is the exact cohomology sequence

$$\ldots \to H^{n-1}(E(m)) \xrightarrow{\delta} H^n(XP(m)) \xrightarrow{j^*} H^n(XP(m-1))$$
$$\xrightarrow{p_m^*} H^n(E(m)) \xrightarrow{\delta} \ldots,$$

j the inclusion of $XP(m-1)$ into $XP(m)$. Since $p_m^*(y_i) = 0$, choose $z_i \in H^*(P(m, X))$ to be such that $j^*(z_i) = y_i$ and $\mathfrak{Z} = \{z_i\}_I$. The element z_i will be called a $(m+1)$-transgression of x_i. Set $N(m) = U(Z/m+1)$.

THEOREM 1. *If $H^*(X)$ is primitively generated and $XP(m)$ is defined, then there exists a trivial k-algebra S such that as k-algebras*

$$H^*(XP(m)) \cong N(m) \oplus S.$$

More specific results are possible if $k = Z(p)$, p a prime. Let J be the ideal of $N(m)$ generated by the odd dimensional elements of $N(m)$ and $\hat{N}(m)$ the subalgebra generated by $\mathfrak{Z}^e$, then $N(m) = \hat{N}(m) \oplus J$. Define $\hat{S} = S \oplus J$.

THEOREM 2. *If $H^*(X)$ is primitively generated, $k = Z(p)$, and $XP(m)$ is defined, then $\hat{S}$ is an $A(p)$-module and there is the vector space isomorphism*

$$H^*(XP(m)) \cong \hat{N}(m) \oplus \hat{S}$$

so that it is possible to induce an $A(p)$-algebra structure on $\hat{N}(m)$.

2. Proof of Theorem 1. Theorem 1 is proved by induction starting with $P(1, X) = SX$, the reduced suspension of X. In [2] it was shown that the primitive elements are the 1-transgressive elements of $H^*(X)$ so that Theorem 1 is immediately satisfied for this case.

Induction hypothesis: $H^*(XP(n-1)) = N' \oplus S'$ as in the theorem, where N' is generated by $\mathcal{Y}' \subseteq H^*(XP(n-1))$, being constructed as $\mathfrak{Z}$ was above. Since the n-transgressive elements are $(n-1)$-transgressive, we may choose $\mathcal{Y}' \supseteq \mathcal{Y}$.

Let D be a subspace of $H^*(X)$ complementary to $T_1 = T$ and let $\mathcal{X}$ be a basis for T. Since $H^q(X)$ is finite dimensional as a vector space for all q, we can choose a dual basis $\mathcal{X}'$ for $T^* \subseteq H_*(X)$ such that if $x_i \in \mathcal{X}$ and $w_j \in \mathcal{X}'$, then $\langle x_i, w_j\rangle$ is 1 if $i = j$ and 0 otherwise.

LEMMA 2.1. *There is an isomorphism* f: $\bigoplus^n \overline{H}^*(X) \to \overline{H}^*(E(n))$ *such that if each* x_i *is primitive and transgresses to* z_i, *then*

$$\begin{aligned}\delta(f(x_1 \otimes \cdots \otimes x)) &= \delta(x_1 * \cdots * x_n)\\ &= \pm z_1 \cup \cdots \cup z_n.\end{aligned}$$

The existence of an isomorphism is known from [7]. The formula is obtained from a direct application of Theorem (2.4) of [15].

Letting $f\sharp$ be the dual of f, $f\sharp$: $\overline{H}_*(E(n)) \to \bigotimes^n H_*(X)$ is also an isomorphism. Now define $S' = f(S_n)$ where $S_2 = (T \otimes D) + (D \otimes T) + (D \otimes D)$ and $S_{i+1} = (T + D) \otimes S_i$. We then define $S = \delta(S')$.

We now show that $S \cap N = 0$ and that the products of less than $n + 1$ elements of Z are linearly independent. Let $z \in S \cap N$. Since $z \in N$, $z = \sum a_{i,j}\bar{z}_{i,j}$ where this denotes a finite sum, $a_{i,j} \in k$, and

$$\bar{z}_{i,j} = z_{i(1)} \cup \cdots \cup z_{i(j)}$$

is the product of j elements of Z. For convenience it is assumed that the cup product $\bar{z}_{i,j}$ is taken in such a manner that the indices are nondecreasing from left to right. Since $z \in X$, there is $s \in S'$ such that $\delta(s) = z$, and therefore, $j^*(z) = 0$. Observe that $XP(n)$ is of category $n + 1$ so that $\bar{z}_{i,m} = 0$ for $m > n$. Now $j^*(z_i) = y_i \in \mathcal{Y}$ so $j^*(z) = \sum a_{i,j}y_{i,j}$. By the induction assumption, $a_{i,j} = 0$ for $j < n$; hence, $z = \sum a_{i,j}\bar{z}_{i,j}$. By Lemma 2.1

$$\bar{z}_{i,n} = \varepsilon_{i,n}\delta(x_{i,n}), \qquad \varepsilon_{i,n} = \pm 1,$$

where $x_{i,n} = x_{i(1)} * \cdots * x_{i(n)}$. Let $a = \sum \varepsilon_{i,n}a_{i,n}x_{i,n} \in H^*(E(n))$ so that $\delta(a) = \delta(s) = z$ and $\delta(a - s) = 0$. Let $c \in H^*(XP(n))$ be such that $p^*(c) = a - s$ and define

$$w_{j,n} = f\sharp^{-1}(w_{j(1)} \otimes \cdots \otimes w_{j(n)}) = w_{j(1)} * \cdots * w_{j(n)}.$$

Note that $\langle x_{i,n}, w_{i,n}\rangle$ is 1 if each $i_k = j_k$ and is 0 otherwise. Let $\alpha = \deg w_{i(2)}$ and $\beta = \deg w_{i(2)}$. If $x' = w_{j,n} - (-1)^{\alpha\beta}w_{i(2)} * w_{i(1)} * \cdots * w_{i(n)}$, then

$$\langle a - s, x'\rangle = \langle a, x'\rangle - \langle s, x'\rangle = \langle a, w_{i,n}\rangle = a_{i,n}.$$

LEMMA 2.2. $p_*(x') = 0$.

Hence, $\langle a - s, x' \rangle = \langle p^*(c), x' \rangle = \langle c, p_*(x') \rangle = 0$ for $i_1 \neq i_2$ so that in this case $a_{i,n} = 0$. If deg $z_{i(1)}$ is odd, $p \neq 2$ and $i_1 = i_2$, then

$$z_{i(1)} \cup z_{i(2)} = -z_{i(1)} \cup z_{i(2)} = 0$$

so that $z_{i,n} = 0$.

LEMMA 2.3. *Let* $i_1 = i_2$. *If* $p \neq 2$ *and* deg $w_{i(1)}$ *is odd, or* $p = 2$, *then* $p_*(w_{i,j}) = 0$.

Hence, $a_{i,n} = 0$ so that $z = 0$.

We next show that $H^*(XP(n)) = N + S$. If $x \in H^*(XP(n))$, then $j^*(z) \in N'$ so $j^*(z) = \sum a_{i,j}u_{i,j}$. Let $a = \sum a_{i,j}z_{i,j} \in N$, then $j^*(z - a) = 0$. Choose $s \in H^*(E(n))$ so that $\delta(s) = z - a$. Now $\delta(H^*(E(n))) \subseteq N + S$ and $a \in N$ so $z \in N + S$.

3. Proof of Theorem 2. Let T^0 be the linear subspace of $H^*(X)$ generated by $\mathfrak{X}^o$ the odd dimensional elements of $\mathfrak{X}$ and define D^0 to be the linear subspace generated by D and $\mathfrak{X}^e$, the even dimensional elements of $\mathfrak{X}$. Define $U_2 = (T^0 \otimes D^0) + (D^0 \otimes T^0) + (D^0 \otimes D^0)$ and $U_{i+1} = (T^0 + D^0) \otimes U_i$. Let $U' = f(U_n)$. If L^0 is the linear subspace of $H^*(XP(n))$ generated by $\mathfrak{Z}^0$, then $\hat{S} = L + \delta(U')$. Notice that since $\hat{A}(p)$ is all in even degrees, $\hat{A}(p)(T^0) \subseteq T^0$. Now $U + (T^0 \otimes \cdots \otimes T^0) = \otimes^n \overline{H}^*(X)$ and $T^0 \otimes \cdots \otimes T^0$ is an $\hat{A}(p)$-module. Hence, U has an $\hat{A}(p)$-module structure and consequently $\delta(f(U))$ is an $A(p)$-module. Since the elements of $\hat{N}$ are all of even degree and L is all in odd degrees, $\hat{A}(p)L \cap N = \{0\}$. Hence, $\hat{S}$ is an $A(p)$-module. By the Cartan formula, $\hat{S}$ is an $\hat{A}(p)$-algebra.

4. Proof of lemmas. The proofs of Lemmas 2.1, 2.2, and 2.3 depend on the definitions of $E(n)$ and $XP(n)$ described in §1.

PROOF OF LEMMA 2.1. Consider the Mayer-Vietoris sequence

$$\cdots H^*(E(n)) \xrightarrow{i} H^*(E(n-1)) \oplus H^*(X) \xrightarrow{j} H^*(E(n-1) \times X) \xrightarrow{\delta} \cdots.$$

The map δ is an epimorphism since $i = 0$. Since Ker $\delta = (H^*(E(n-1)) \otimes k) + (k \otimes H^*(X)) \subseteq H^*(E(n-1)\Lambda X)$, it is immediate that

$$f_n = \delta\eta^*: H^*(E(n-1)\Lambda X) \to H^*(E(n))$$

is an isomorphism where $\eta: X \times X \to X \Lambda X$ is the quotient map. If for $n = 2$, $f = f_2$ and for $n = k + 1$, $f = f_{k+1} \circ (f_k \otimes 1)$, then f will be an isomorphism for all n.

If $w \in H^*(E(n-1))$ and $x \in H^*(X)$, then $w * x = \delta(w \otimes x) \in H^*(E(n))$ as defined in [13]. By the way f_n was constructed, $f_n(w \otimes x) = w * x$. Hence, we conclude that

$$f(x_{i(1)} \otimes \cdots \otimes x_{i(n)}) = x_{i(1)} * \cdots * x_{i(n)}$$

as required.

Proof of Lemma 2.2. For $n = 2$ we have the Hopf construction p: $X \circ X \to SX$. Let $(X \circ X, E, X')$ and (SX, M, C) be the usual decompositions of these spaces, so that p is a map of these triples. Recall that $p|X \times X = m$, the H-space multiplication. Now consider the diagram below

$$\begin{tikzcd}
H_*(X) \otimes H_*(X) \cong H_*(X \wedge X) \arrow[dr, "\eta_*"] \arrow[ddr, "m\#"'] & & \bar{H}_*(X) \otimes \bar{H}_*(X) \arrow[ll] \arrow[d, "f\#^{-1}"] \\
& H_*(X \times X) \arrow[r, "\delta"] \arrow[d, "m_*"] & H_*(X \circ X) \arrow[d, "p_*"] \\
& H_*(X) \arrow[r, dash, "\sigma"] & H_*(SX)
\end{tikzcd}$$

where $m_\#$ is the Pontrjagin product. Let $w_{i(1)}$ and $w_{i(2)} \in \bar{H}_*(X)$, then since the diagram is commutative,

$$p_*(x') = \sigma\big(m_\#(w_{i(1)}, w_{i(2)}) - (-1)^{\deg(w_{i(1)})\deg(w_{i(2)})} m_\#(w_{i(1)}, w_{i(2)})\big).$$

Now since $H^*(X)$ is primitively generated, $H_*(X)$ has a commutative Pontrjagin product, [**8**], and $p_*(w) = 0$.

The proof can be completed by an induction argument using the fact that p is a map of triples.

Proof of Lemma 2.3. This is an induction argument very much like the one above. Let $w = w_{i(1)} = w_{i(2)}$. By the proof of Lemma 2.2, $p_*(w * w) = w^2$. If the characteristic is 2, then $w^2 = 0$ since $H_*(X)$ has a commutative Pontrjagin product. If the characteristic is not 2, then w is odd dimensional so that $w^2 = - w^2 = 0$. Using the decomposition of $E(n)$ and $P(n, X)$ mentioned above, an induction argument completes the proof of this lemma.

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