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PERIODIC AND FIXED POINTS, AND COMMUTING MAPPINGS

GERALD JUNGCK

ABSTRACT. We employ commuting mappings to produce generalizations of locally contractive and locally expansive maps, and obtain criteria for the existence of fixed and periodic points of arbitrary maps on compacta.

1. Introduction. Our main result cites conditions which ensure that a map of a compact metric space onto itself has periodic points. This one theorem yields as by-products a necessary and sufficient criterion for the existence of fixed points (Corollary 2.3), a generalization of results of Bailey **[1]** and Holmes **[3]** on locally contractive maps, and a theorem on periodic points for open and generalized locally expansive maps.

By the term map we shall mean a continuous function. A map $f: (X, d) \to (X, d)$ is locally contractive (expansive) iff there exists $\varepsilon > 0$ such that $d(f(x), f(y)) < d(x, y)$ ($> d(x, y)$) whenever $0 < d(x, y) < \varepsilon$. Maps f, g: $X \to X$ commute iff $fg = gf$. If $g: X \to X$, we let C_g denote the set of all maps $f: X \to X$ which commute with g. In addition, N denotes the set of natural numbers, and for each $n \in N$, f^n denotes the nth composite of f.

The fact that a function $f: X \to X$ has fixed points iff there is a constant function $g: X \to X$ which commutes with f prompts the investigation of commuting mappings in the search for fixed points (see **[4]**). Their potential as a means of generalizing lies in the fact that, for any n, $f^n \in C_g$ if $f \in C_g$ (in particular, $g^n \in C_g$).

2. Main result and corollaries.

THEOREM 2.1. *Let g be a surjective map of a compact metric space (X, d) to itself. Suppose ε is a positive number satisfying the condition: if $x,y \in X$ and $0 < d(g(x), g(y)) < \varepsilon$ then there is at least one $f \in C_g$ and $z \in g^{-1}(g(x))$ such that*

$$d(f(z), f(y)) < d(g(x), g(y)).$$

Then g has a periodic point. In fact, if k is a positive integer such that, for some x, $d(x, g^k(x)) < \varepsilon$, then $a = g^k(a)$ for some $a \in X$.

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Proof. Let $x \in X$. The sequence $\{g^n(x)\}$ has a convergent subsequence since (X, d) is compact. Specifically, there exist k and m such that

$$d(g^m(x), g^{m+k}(x)) = d(g^m(x), g^k(g^m(x))) < \varepsilon.$$

Thus:

(i) $d(y, g^k(y)) < \varepsilon$ for some $y \in X$.

Suppose that $k \in N$ for which (i) holds. Since the composite $g^k\colon X \to X$ is continuous and (X, d) is compact, there exists $a \in X$ for which

(ii) $d(a, g^k(a)) \leqslant d(x, g^k(x))$ for all $x \in X$.

We assert that $a = g^k(a)$. Otherwise, (i) implies $0 < d(a, g^k(a)) < \varepsilon$. But g is a surjection, so that $g(c) = a$ for some $c \in X$, and we have

$$0 < d(g(c), g^k(g(c))) = d(g(c), g(g^k(c))) < \varepsilon.$$

Consequently, the hypothesis yields $f \in C_g$ and $z \in g^{-1}(g(c))$ such that

$$d(f(z), f(g^k(c))) < d(g(c), g(g^k(c))).$$

Since $z \in g^{-1}(g(c))$, $g^k(z) = g^k(c)$, and hence

$$d(f(z), f(g^k(z))) < d(g(c), g(g^k(c))).$$

But then

$$d(f(z), g^k(f(z))) < d(a, g^k(a))$$

since $f \in C_g$. This last inequality contradicts (ii). □

Now let g be any map (not necessarily surjective) of a compact metric space X to itself. If $A = \bigcap_{n=1}^{\infty} g^n(X)$, then A is compact, $g(A) = A$, and if $f \in C_g$, then $f|_A \in C_g|_A$. Thus, if we require that the $z \in g^{-1}(g(x))$ in Theorem 2.1 be x, we can apply 2.1 to $g|_A\colon A \to A$ to conclude:

Corollary 2.2. *Let g be a map of a compact metric space (X, d) to itself. If ε is a positive number such that whenever $0 < d(g(x), g(y)) < \varepsilon$, there exists $f \in C_g$ for which*

$$d(f(x), f(y)) < d(g(x), g(y)),$$

then g has a periodic point. In fact, if k is a positive integer such that, for some $x \in A = \bigcap_{n=1}^{\infty} g^n(X)$, $d(x, g^k(x)) < \varepsilon$, then $a = g^k(a)$ for some $a \in X$.

The following fixed point theorem is stated without proof in [4].

Corollary 2.3. *A map g of a compact metric space (X, d) into itself has a fixed point iff $g(x) \neq g(y)$ implies there is some $f \in C_g$ such that $d(f(x), f(y)) < d(g(x), g(y))$.*

Proof. The necessity portion is easily proved by considering the constant function $f(x) \equiv a$, where a is a fixed point of g (see the proof of the theorem in [4]). To prove "sufficiency", let $\varepsilon = 2\,\mathrm{diam}(X)$, note that $d(x, g(x)) < \varepsilon$ for all $x \in X$, and appeal to Corollary 2.2.

3. Locally contractive maps generalized.

THEOREM 3.1. *Let H be a commutative semigroup of maps of a compact metric space (X, d) to itself and let $g \in H$. If ε is a positive number such that whenever $0 < d(g(x), g(y)) < \varepsilon$, there is some $h \in H$ satisfying*

$$d(h(x), h(y)) < d(g(x), g(y)),$$

then g and any finite subfamily of H have a common periodic point.

PROOF. For $h \in H$ and $k \in N$ let $P(h, k) = \{x \in X: h^k(x) = x\}$. These sets are closed since h is continuous. Note also that $f(P(h, k)) \subset P(h, k)$ if $f \in H$, since f and h commute. By Corollary 2.2 there exists m such that $P(g, m) \neq \varnothing$. Denote $P(g, m)$ by P_g, and consider:

LEMMA. *Assume the hypothesis of Theorem 3.1, and let F be a nonempty closed subset of P_g such that $h(F) \subset F$ for all $h \in H$. If $h \in H$, h has a periodic point in F.*

PROOF. Since P_g is closed, (F, d) is compact. Let $h \in H$ and apply the initial argument of the proof of Theorem 2.1 to $h|_F$ to produce $a \in F$ and $k \in N$ such that $d(a, h^k(a)) < \varepsilon$ and

$$d(a, h^k(a)) \leq d(x, h^k(x)) \quad \text{for } x \in F. \tag{$*$}$$

Suppose $a \neq h^k(a)$. Since $a \in P_g$, $g^m(a) = a$. Let $c = g^{m-1}(a)$. Then

$$d(h^k(a), a) = d(h^k(g(c)), g(c)) = d(g(h^k(c)), g(c)).$$

Thus $0 < d(g(h^k(c)), g(c)) < \varepsilon$, so by hypothesis there exists $f \in H$ such that

$$d(g(h^k(c)), g(c)) > d(f(h^k(c)), f(c));$$

i.e., $d(h^k(a), a) > d(h^k(f(c)), f(c))$. But $f(c) \in F$ since $a \in F$ and hence $c \in F$; consequently, the last inequality above denies $(*)$. Conclude that $a = h^k(a)$. □

We now complete the proof of Theorem 3.1. Observe that P_g is a nonempty closed subset of P_g such that $h(P_g) \subset P_g$ for $h \in H$. The Lemma therefore yields $k = k(h)$ for each $h \in H$ such that $P_g \cap P(h, k) \neq \varnothing$. Thus any singleton subset of H has a periodic point in common with g. We proceed by induction.

Let $\{h_1, h_2, \ldots, h_n, h_{n+1}\} \subset H$ and assume that

$$G_n = P_g \cap \left(\bigcap_{i=1}^{n} P(h_i, k_i) \right) \neq \varnothing.$$

Since $f(P(h, k)) \subset P(h, k)$ for any $f, h \in H$ and $k \in N$,

$$f(G_n) \subset f(P_g) \cap \left(\bigcap_{i=1}^{n} f(P(h_i, k_i)) \right) \subset G_n \quad \text{for any } f \in H.$$

Applying the Lemma to h_{n+1} and G_n guarantees the existence of a $k_{n+1} \in N$ such that $G_n \cap P(h_{n+1}, k_{n+1}) \neq \varnothing$. Thus, by induction, g and any finite subset of H have a common periodic point. □

Suppose that in Theorem 3.1 we require that $d(f(x), f(y)) < d(g(x), g(y))$ for at least one $f \in H$ whenever $g(x) \neq g(y)$. We know by Corollary 2.3 above that g has a fixed point ($P(g, 1) \neq \varnothing$). A modification of the above proof and Lemma shows that under the new hypothesis the set $\{P(g, 1) \cap P(h, 1): h \in H\}$ has the finite intersection property, and hence has a non-empty intersection by compactness. In fact, there is a unique(!) point $a \in X$ such that $a = g(a) = h(a)$ for *all* $h \in H$.

Moreover, Theorem 3.1 is a generalization of the following theorem by Holmes, and hence of Bailey's Corollary 2 to Theorem 2 in **[1]**.

THEOREM (HOLMES [3]). *Let (X, d) be compact and C a commutative semigroup of maps $f: X \to X$. Suppose there exist $\lambda \in (0, 1)$ and $\varepsilon > 0$ such that*

(i) *$d(x, y) < \varepsilon$ implies there exists $n(x, y) \in I^+$ and $f_{x,y} \in C$ for which $d(f^n(x), f^n(y)) \leq \lambda d(x, y)$. Then each finite subfamily of C has a common periodic point.*

Now since by definition, a semigroup is closed with respect to the given operation, we know $f^n \in C$ when $f \in C$ in Holmes' Theorem. Consequently, (i) can be simplified to read:

(ii) *$d(x, y) < \varepsilon$ implies there exists $f_{x,y} \in C$ for which $d(f(x), f(y)) \leq \lambda d(x, y)$.*

Thus it is clear that Holmes' result follows from our Theorem 3.1 with $g = i$ ($i(x) = x$ for $x \in X$) and $H = C \cup \{i\}$. The following simple example demonstrates that Theorem 3.1 indeed generalizes Holmes' result.

EXAMPLE. Let $X = [0, 1]$ and $d(x, y) = |x - y|$. Define $f: X \to X$ by

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then

$$f^n(x) = \begin{cases} 2^n x, & \text{if } 0 \leq x \leq 2^{-n}, \\ 1, & \text{if } 2^{-n} \leq x \leq 1. \end{cases}$$

Let $C = H = \{f^n: n \in N \cup \{0\}\}$, where $f^0 = i$.

Then $d(f^n(0), f^n(x)) \geq d(0, x)$ for any $x \in [0, 1]$ and any $f^n \in C$, so the hypothesis of Holmes' Theorem is not satisfied. On the other hand, let $\varepsilon = \frac{1}{4}$ and $g = f$ in the statement of our Theorem 3.1. Let $x, y \in [0, 1]$ and choose notation so that $x < y$. Then if $y \leq \frac{1}{2}$,

$$d(g(x), g(y)) = 2|x - y| > d(x, y),$$

and we let $h = i$. If $y > \frac{1}{2}$, $g(y) = 1$. Thus if $0 < d(g(x), g(y)) < \varepsilon$, $0 < g(x) = 2x < 1$ and $1 - 2x < \frac{1}{4}$, so that $x > 3/8$. Consequently, $f^2(x) = f^2(y) = 1$, and we have $d(g(x), g(y)) > 0 = d(f^2(x), f^2(y))$. In this instance let $h = f^2$. In any event, we can choose h so that the hypothesis of Theorem 3.1 is satisfied.

4. Locally expansive maps generalized. A map $f\colon (X, d) \to (X, d)$ is an expansion if there exists a real $r > 1$ such that $d(f(x), f(y)) \geqslant rd(x, y)$. In [2] Borges considered continuous expansions on complete metric spaces, and intimated that further study of expansions is in order. Rosenholtz studied *local* expansions in [6] and locally expansive maps in [7]. In [7] he proved that any open locally expansive map of a *connected* compact metric space into itself has a fixed point. We prove:

THEOREM 4.1. *Let g be an open map of a compact metric space (X, d) onto itself. If there exists $\varepsilon > 0$ such that $0 < d(x, y) < \varepsilon$ implies that for at least one $f \in C_g$, $d(f(x), f(y)) < d(g(x), g(y))$, then g has a periodic point.*

PROOF. By hypothesis, g is actually a local homeomorphism of a compact metric space onto itself, and hence a covering projection. In fact (see [6]), each $a \in X$ has a neighborhood V_a such that $g^{-1}(V_a)$ is the union of finitely many mutually disjoint open sets U_a such that $\operatorname{diam}(U_a) < \varepsilon$, and $g(U_a) = V_a$. Let λ be the Lebesgue number of the open covering $\{V_a\colon a \in X\}$.

By Theorem 2.1 it suffices to prove:

$$0 < d(g(x), g(y)) < \lambda \quad \text{implies} \quad d(g(x), g(y)) > d(f(z), f(y)) \qquad (**)$$

for some $z \in g^{-1}(x)$ and some $f \in C_g$.

To this end, suppose $0 < d(g(x), g(y)) < \lambda$. By the definition of λ we can choose $a \in X$ such that $g(x), g(y) \in V_a$, and hence an open set U_a such that $\operatorname{diam}(U_a) < \varepsilon$, $g(U_a) = V_a$, and $y \in U_a$. Since $g(U_a) = V_a$, $g(x) = g(z)$ for some $z \in U_a$. But $z, y \in U_a$ implies $d(z, y) < \varepsilon$; moreover, $0 < d(z, y)$ since $g(x) \neq g(y)$. The hypothesis then yields $f \in C_g$ for which

$$d(f(z), f(y)) < d(g(z), g(y)) = d(g(x), g(y)),$$

and (**) holds. $\square$

(Compare Theorem 4.1 to Theorem 1 in [5].)

COROLLARY. *Let g be an open map of a compact metric space (X, d) onto itself. If there exists $\varepsilon > 0$ such that*

$$0 < d(x, y) < \varepsilon \quad \textit{implies} \quad d(g^n(x), g^n(y)) < d(g(x), g(y))$$

for at least one nonnegative integer $n = n(x, y)$, then g has a periodic point.

Note that with $n = n(x, y) = 0$ in the above corollary, we obtain the result that any locally expansive open map of a compact metric space onto itself has a periodic point.

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Department of Mathematics, Bradley University, Peoria, Illinois 61625