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EDELSTEIN'S CONTRACTIVITY AND ATTRACTORS

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ABSTRACT. In this article an example is constructed to show that Theorem 1.1 of L. Janos [Canad. Math. Bull. **18** (1975), no. 5, 675–678] is false. A proper formulation is obtained as follows. Theorem. If (X, τ) is a metrizable topological space, f: $X \to X$ is continuous, and $a \in X$, then the following statements are equivalent:

(1) There exists a metric d compatible with τ such that f is contractive with respect to d and the sequence $(f^n(x))_{n=1}^{\infty}$ converges to a for every $x \in X$.

(2) The singleton $\{a\}$ is an attractor for compact subsets under f.

Furthermore, under this proper formulation, we show that Theorem 3.2 Janos [Proc. Amer. Math. Soc. **61** (1976), 161–175] and Theorem 2.3 Janos and J. L. Solomon [ibid. **71** (1978), 257–262], where the false Theorem 1.1 in [2] has been quoted in the original proofs, remain valid.

1. Introduction. In recent years several authors tried to characterize different kinds of contractivity of self-maps f: $X \to X$ on a metric space (X, d). Since the hypotheses of a fixed point theorem for f usually contain conditions of different natures such as

(a) topological properties of X,

(b) metric properties of (X, d),

(c) topological properties of f,

(d) metric properties of f,

it may be of interest to separate those conditions which are purely topological in nature from those which are metric dependent. In [2], L. Janos attempted to give such a characterization to a fixed point theorem of Edelstein [1]. In this paper we first show that the main theorem (Theorem 1.1) in [2] is unfortunately false by providing a counterexample in §2. Next by employing the elegant concept of attractor, we offer in §3 a simple and nicely quotable result illuminating the link between the metric dependent notion of Edelstein contractivity and topological notion of attractor, and thus we provide a proper formulation of Theorem 1.1 in [2]. As Theorem 1.1 has been used in

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showing Theorem 3.2 in [3] and Theorem 2.3 in [4], we show, in §4, that under our new formulation, these two theorems remain valid.

2. Preliminaries and a counterexample.

DEFINITION 1. Let (X, d) be a metric space and $f: X \to X$. Then

(i) f is nonexpansive with respect to d if $d(f(x), f(y)) \leqslant d(x, y)$, $\forall x, y \in X$.

(ii) f is Edelstein contractive [1] (or simply contractive if no confusion arises) with respect to d if $x \neq y$ implies $d(f(x), f(y)) < d(x, y)$.

The following definition was first introduced by R. Nussbaum in [6].

DEFINITION 2. Let X be a topological space and $f: X \to X$. Then a subset A of X is an attractor for compact sets under f if

(1) A is nonempty compact and f-invariant, and

(2) for any open set G containing A, and any compact set K in X, there exists a positive integer N such that $f^n(K) \subset U$, $\forall n \geqslant N$.

If (X, τ) is a metrizable space, we shall denote by $M(\tau)$ the family of all metrics on X which generate the topology τ on X.

In [2, Theorem 1.1] L. Janos proclaimed the following:

$(*)$ Let (X, τ) be a metrizable topological space, $f: X \to X$ be continuous such that the sequence $(f^n(x))_{n=1}^{\infty}$ converges for every $x \in X$. Then the following two statements are equivalent:

(i) There is d in $M(\tau)$ such that f is contractive relative to d.

(ii) For every nonempty compact f-invariant subset Y of X the intersection of all iterates $f^n(Y)$ is a one point set.

The following example shows that the statement $(*)$ is, in fact, false.

EXAMPLE. Let $X = \{(0, 0)\} \cup \{(1/n, m/n): m = 0, 1, 2, \ldots, n^2, n = 1, 2, \ldots\}$ be equipped with the (relative) usual topology τ. Define $f: X \to X$ by

$$f\left(\frac{1}{n}, n\right) = f(0, 0) = (0, 0), \quad \text{for } n = 1, 2, \ldots,$$

$$f\left(\frac{1}{n}, \frac{m}{n}\right) = \left(\frac{1}{n}, \frac{m+1}{n}\right), \quad \text{for } m = 0, 1, \ldots, n^2 - 1, \quad n = 1, 2, \ldots.$$

It is readily seen that (a) f is continuous on X, (b) for each $x \in X$, the sequence $(f^n(x))_{n=1}^{\infty}$ converges to the unique fixed point $a = (0, 0)$, and (c) $\bigcap_{n=0}^{\infty} f^n(X) = \{a\}$. From (c) it follows that f satisfies (ii) of statement $(*)$. We shall now show that f does not satisfy (i) of statement $(*)$. Indeed if there were a metric $d \in M(\tau)$ such that f is contractive w.r.t. d, then f is nonexpansive w.r.t. d. Since $U = \{x \in X: |x - a| < \frac{1}{2}\}$ is an open neighbourhood of a, there exists $\varepsilon > 0$ such that $B = \{x \in X: d(x, a) < \varepsilon\} \subset U$. But then $f^n(B) \subset B \subset U$ for every $n = 1, 2, \ldots$. On the other hand, there must exist a positive integer n and a nonnegative integer m such that $x = (1/n, m/n) \in B$. Note that $n \geqslant 2$ and $m < n/2$ as $B \subset U$. It follows that $f^{n-m}(x) = (1/n, 1) \notin U$ which is a contradiction.

Our effort is thus to give a correct and proper formulation of the statement $(*)$.

3. Characterization of contractivity via attractor.

THEOREM. *Let (X, τ) be a metrizable topological space, let f: $X \to X$ be continuous, and let $a \in X$. Then the following statements are equivalent*:

(1) *There exists d in $M(\tau)$ such that f is contractive with respect to d and the sequence $(f^n(x))_{n=1}^{\infty}$ converges to a for every $x \in X$;*

(2) *The singleton $\{a\}$ is an attractor for compact subsets under f.*

Furthermore if (X, τ) is topologically complete, then the metric d in (1) *can be chosen to be complete.*

PROOF. (1) $\Rightarrow$ (2). Suppose that the singleton $\{a\}$ is not an attractor for compact sets under f; then there is a nonempty compact subset C of X and an open set U containing a such that for any positive integer n, $\exists$ integer $m \geqslant n$ such that $f^m(C) \not\subset U$. Thus we can choose an increasing sequence $(n_i)_{i=1}^{\infty}$ of positive integers and points $x_i \in C$ such that $f^{n_i}(x_i) \notin U$ for $i = 1, 2, \ldots$. Since C is compact, by passing to a subsequence if necessary, we may assume $(x_i)_{i=1}^{\infty}$ converges to $x_0 \in C$. Let $d \in M(\tau)$ satisfy (1). Then for arbitrary $\varepsilon > 0$, there exist N_1, N_2 such that

$$i \geqslant N_1 \Rightarrow d\big(f^i(x_0), a\big) < \varepsilon/2$$

and

$$i \geqslant N_2 \Rightarrow d(x_i, x_0) < \varepsilon/2.$$

Let $N_3 = \max\{N_1, N_2\}$. Then

$$\begin{aligned} i \geqslant N_3 \Rightarrow d(f^{n_i}(x_i), a) &\leqslant d(f^{n_i}(x_i), f^{n_i}(x_0)) + d(f^{n_i}(x_0), a) \\ &\leqslant d(x_i, x_0) + d(f^{n_i}(x_0), a) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This shows that the sequence $(f^{n_i}(x_i))_{i=1}^{\infty}$ converges to a which contradicts the assumptions that $f^{n_i}(x_i) \notin U$, $\forall i$, and U is a neighbourhood of a. Hence $\{a\}$ is an attractor for compact subsets under f.

(2) $\Rightarrow$ (1). First we observe that $(f^n(x))_{n=1}^{\infty}$ converges to a for all $x \in X$, since $\{a\}$ is an attractor for compact sets under f. Let $d \in M(\tau)$ (choose a complete metric d in $M(\tau)$ if X is topologically complete). Define

$$d^*(x, y) = \sup\{d(f^n(x), f^n(y)) \colon n = 0, 1, 2, \ldots\}, \quad \forall x, y \in X.$$

d^* is well defined since $f^n(x) \to a$, $\forall x \in X$. One can easily prove that d^* is a metric on X such that $d^* \geqslant d$ and that f is nonexpansive w.r.t. d^*. To show that $d^* \in M(\tau)$, that is to show d and d^* are equivalent, it suffices to show, for any sequence $(x_n)_{n=1}^{\infty}$ in X and $x_0 \in X$, $d(x_n, x_0) \to 0$ as $n \to \infty \Rightarrow d^*(x_n, x_0) \to 0$ as $n \to \infty$. Suppose on the contrary that $d(x_n, x_0) \to 0$ but $d^*(x_n, x_0)$ does not converge to 0. By passing to a subsequence, we may assume without loss of generality that for some $\varepsilon > 0$

$$d^*(x_n, x_0) \geqslant \varepsilon \quad \text{for all } n = 1, 2, \ldots. \tag{†}$$

For each n, $\exists$ a positive integer $k(n)$ such that

$$d^*(x_n, x_0) = d\big(f^{k(n)}(x_n), f^{k(n)}(x_0)\big).$$

Let $A = \{k(n): n = 1, 2, \ldots\}$.

Case 1. Suppose the set A is a finite set. Then there exists a positive integer k and a subsequence $(k(n_i))_{i=1}^{\infty}$ of $(k(n))_{n=1}^{\infty}$ such that $k(n_i) = k$, for $i = 1, 2, \ldots$. As f is continuous, we have

$$d^*(x_{n_i}, x_0) = d\big(f^{k(n_i)}(x_{n_i}), f^{k(n_i)}(x_0)\big) = d\big(f^k(x_{n_i}), f^k(x_0)\big) \to 0 \quad \text{as } i \to \infty.$$

This contradicts (†).

Case 2. Suppose the set A is an infinite set. Then we can extract a strictly increasing subsequence $(k(n_i))_{i=1}^{\infty}$ from A. Let $C = \{x_n: n = 1, 2, \ldots\} \cup \{x_0\}$. Then C is a compact set. As $\{a\}$ is an attractor for compact sets under f, then for this ε, $\exists$ a positive integer m such that

$$n > m \Rightarrow f^n(C) \subset B(a, \varepsilon/2) = \{y \in X: d(y, a) < \varepsilon/2\}.$$

Let i_0 be such that $i > i_0 \Rightarrow k(n_i) > m$. Then

$$\begin{aligned} i > i_0 \Rightarrow d^*(x_{n_i}, x_0) &= d\big(f^{k(n_i)}(x_{n_i}), f^{k(n_i)}(x_0)\big) \\ &< d\big(f^{k(n_i)}(x_{n_i}), a\big) + d\big(f^{k(n_i)}(x_0), a\big) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This again contradicts (†).

Therefore d^* and d are equivalent. To produce a metric satisfying (1), let

$$d^{**}(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} d^*(f^n(x), f^n(y)), \quad \forall x, y \in X.$$

Then d^{**} is a metric on X such that f is nonexpansive w.r.t. d^{**}. As $d^* < d^{**} < 2d^*$, we see that $d^{**} \in M(\tau)$. To prove that f is contractive w.r.t. d^{**}, assume that $x \neq y$ and $d^{**}(f(x), f(y)) = d^{**}(x, y)$. Then from the nonexpansiveness of f w.r.t. d^{**} this implies that

$$d^*(f^{n+1}(x), f^{n+1}(y)) = d^*(f^n(x), f^n(y)), \quad \forall n = 0, 1, 2, \ldots.$$

That is,

$$d^*(f^n(x), f^n(y)) = d^*(x, y) \neq 0, \quad \forall n = 1, 2, \ldots.$$

This contradicts the fact that $d^*(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ (because $f^n(x) \to a$, $\forall x \in X$). Hence f is contractive w.r.t. d^{**}. Furthermore, from the facts that $d < d^* < d^{**}$ and that d, d^* and d^{**} are equivalent, we see that if d is a complete metric, then both d^* and d^{**} will be complete. This completes the proof.

Remark 1. From the proof of (2) $\Rightarrow$ (1), we see that if d is bounded, then the metric d^{**} so constructed is also bounded.

Remark 2. From the proof of (1) $\Rightarrow$ (2), we in fact show that the following statement (3) implies (2) in the above theorem:

(3) There exists $d \in M(\tau)$ such that f is nonexpansive w.r.t. d and the sequence $(f^n(x))_{n=1}^{\infty}$ converges to a for every $x \in X$.

Since clearly (1) implies (3), the conditions (1), (2) and (3) are equivalent under the assumptions of the above theorem.

4. Corollaries. Let (X, τ) be a metrizable topological space, let $f\colon X \to X$ be continuous, and let $M \subset X$ be a nonempty compact, f-invariant set. Let X/M be the quotient space equipped with the quotient topology τ^* arising from X by identifying M with a point, and let $\pi\colon X \to X/M$ be the natural projection and $f^*\colon X/M \to X/M$ be the induced map of f such that $\pi \circ f = f^* \circ \pi$. The following results can be found in [4]:

PROPOSITION 1 (LEMMA 2.2. IN [4]). *$(X/M, \tau^*)$ is metrizable.*

PROPOSITION 2 (THEOREM 2.1. IN [4]). *M is an attractor for compact sets under f if and only if $\{a^*\}$, with $a^* = \pi(M)$, is an attractor for compact sets under f^*.*

From Propositions 1 and 2, the following result (Theorem 2.3 in [4]) is an immediate consequence of our theorem in §3 (its proof was originally based on the false result Theorem 1.1 in [2]).

COROLLARY 1. *Let (X, τ) be a metrizable topological space, and let $f\colon X \to X$ be continuous. If $M \subset X$ is an attractor for compact sets under f, then there exists a metric d^*, compatible with the topology τ^* on X/M, such that f^* is contractive w.r.t. d^*.*

Denote by $\alpha(Y)$ the Kuratowski measure of noncompactness of a subset Y of a bounded metric space (X, d) (see [5] and [7]). We say that $f\colon X \to X$ is condensing if f is continuous and for any nonempty nontotally bounded subset Y of X, $\alpha(f(Y)) < \alpha(Y)$. The following result is contained in the proof of Theorem 3.2 in [3]:

PROPOSITION 3. *Let (X, d) be a bounded complete metric space, and let $f\colon X \to X$ be condensing such that*

$$d(f(x), f(y)) < \tfrac{1}{2}\{d(x, f(x)) + d(y, f(y))\}$$

whenever $x, y \in X$ and $x \neq y$. Then

(i) f has a unique fixed point $a \in X$ such that $f^n(x) \to a$ for every $x \in X$, and

(ii) for every nonempty compact f-invariant subset Y of X, $\bigcap_{n=0}^{\infty} f^n(Y) = \{a\}$.

We shall now be able to show below that, even though the original proof of Theorem 3.2 in [3] was based on the false Theorem 1.1 in [2], its strengthened conclusion remains valid by applying our theorem in §3.

COROLLARY 2. *Let (X, d) be a bounded complete metric space, and let $f\colon X \to X$ be condensing such that*

$$d(f(x), f(y)) < \tfrac{1}{2}\{d(x, f(x)) + d(y, f(y))\}$$

whenever $x, y \in X$ with $x \neq y$. Then

(i) f has a unique fixed point $a \in X$ such that $f^n(x) \to a$ for every $x \in X$, and

(ii) there exists a bounded complete metric d^* on X which is equivalent to d such that f is contractive w.r.t. d^*.

PROOF. By Proposition 3, we need only to show that (ii) holds. To this end, we prove that $\{a\}$ is an attractor for compact sets under f. Let C be any nonempty compact subset of X. Since $\alpha(C \cup B) = \alpha(B)$ for any $B \subset X$, we conclude that

$$\alpha\left(\bigcup_{n=0}^{\infty} f^n(C)\right) = \alpha\left(C \cup \bigcup_{n=1}^{\infty} f^n(C)\right)$$
$$= \alpha\left(\bigcup_{n=1}^{\infty} f^n(C)\right) = \alpha\left(f\left(\bigcup_{n=0}^{\infty} f^n(C)\right)\right).$$

Thus $\bigcup_{n=0}^{\infty} f^n(C)$ is totally bounded since f is condensing. Therefore $Y = \overline{\bigcup_{n=0}^{\infty} f^n(C)}$ is compact and f-invariant, since (X, d) is complete. By Proposition 3 (ii) $\bigcap_{n=0}^{\infty} f^n(Y) = \{a\}$. Since $f^n(Y) \supset f^{n+1}(Y)$ for $n = 0,1,2,\ldots$ and each $f^n(Y)$ is compact, given any open neighbourhood U of a, there exists a positive integer N such that $f^N(Y) \subset U$. It follows that

$$n > N \Rightarrow f^n(C) \subset f^n(Y) \subset f^N(Y) \subset U.$$

This shows that $\{a\}$ is an attractor for compact sets under f. Thus (ii) follows from our Theorem and Remark 1 in §3.

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