

THERE IS NO UNIVERSAL SEPARABLE MOORE SPACE

ERIC K. VAN DOUWEN¹

ABSTRACT. There is no Hausdorff space of cardinality c (in particular, there is no separable Moore space) which includes a homeomorph of every separable Moore space.

A *universal* space for a class \mathfrak{R} of topological spaces is a space U in \mathfrak{R} in which every space belonging to \mathfrak{R} can be embedded. A well-known example is that the Hilbert cube is a universal space for the class of all separable metrizable spaces. This suggests the question, asked by B. Fitzpatrick, in his invited hour address at the November 1973 meeting of the AMS, in Atlanta, Georgia (this information was supplied by Mike Reed), and repeated by Reed, [R, 4, (5)], of whether there is a universal separable Moore space. The following simple negative answer has apparently been overlooked.

We first note that a separable Moore space (indeed, a separable first countable Hausdorff space) has cardinality at most c . Consequently the following is much more than we need.

THEOREM. *For every Hausdorff space X of cardinality (at most) c there is a separable Moore space which cannot be embedded into X .*

The easiest way to prove this would be to show that there are more than 2^c pairwise nonhomeomorphic separable Moore spaces. However, this is not the case. [There are only 2^c pairwise nonhomeomorphic separable regular spaces of cardinality at most c]. So we find it amusing that we prove the theorem by showing that there are too many separable Moore spaces.

For convenience we call a space X ψ -like if it has a countably infinite dense subset D which is what is called *relatively compact*: every infinite subset of D has a cluster point in X . Mrówka has pointed out in [M₂, 3.5] that the space commonly called Ψ , [GJ, 5I], which he introduced in [M₁], has 2^c pairwise nonhomeomorphic variations. Since it is well known (and easy to see) that Ψ is a Moore space, this proves part (a) of the following Lemma, which clarifies our claim that there are too many separable Moore spaces, and implies our Theorem.

Received by the editors January 11, 1979.

AMS (MOS) subject classifications (1970). Primary 54C25, 54E30.

Key words and phrases. Universal, separable Moore space.

¹Research supported by NSF Grant MCS 78-09484.

LEMMA. (a) *There are 2^c pairwise nonhomeomorphic ψ -like Moore spaces.* (b) *if X is a Hausdorff space, then X has at most $|X|^\omega$ first countable ψ -like subspaces.*

PROOF OF (b). It suffices to show that if Y and Z are first countable subspaces of X which have a countable dense subset D in common which is relatively compact in both, then $Y = Z$. If $y \in Y - D$ then some sequence in D converges to y . This sequence must have a cluster point in Z , but y is its only cluster point. Hence $Y \subseteq Z$, and $Z \subseteq Y$ by symmetry. \square

REFERENCES

- [GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.
[M₁] S. Mrówka, *On completely regular spaces*, Fund. Math. **41** (1954), 105–106.
[M₂] ———, *Some set-theoretic constructions in topology*, Fund. Math. **94** (1977), 83–92.
[R] G. M. Reed, *On subspaces of separable first-countable T_2 -spaces*, Fund. Math. **91** (1976), 189–202.

INSTITUTE FOR MEDICINE AND MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701