PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 2, September 1979

# ON THE EXISTENCE OF CLOSED HERMITIAN RESTRICTIONS OF SELFADJOINT OPERATORS

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Von Neumann [3, Satz 48] proved, by examining "the pathology of unbounded operators", that a closed hermitian operator $H$ on a Hilbert space $\mathcal{H}$ has a closed hermitian restriction if and only if it is unbounded.

Hamburger [2, §9] observed that von Neumann's argument did not show how to construct all such restrictions and that nothing was asserted about the deficiency index of the restriction. He then (loc. cit, Theorems 7 and 8) showed how to obtain all such restrictions, of deficiency index $(m : m)$, with $m$ finite or infinite, when $H$ is selfadjoint (and unbounded, as appears in application of his method).

We give here a brief proof of von Neumann's result for the selfadjoint case:

THEOREM. *A selfadjoint operator has a closed hermitian restriction if and only if it is unbounded* (*and then has such a restriction of deficiency index* $(1 : 1)$).

PROOF. The selfadjoint operator $H$ has a closed hermitian restriction $K$ if and only if its Cayley transform $U_H$ (which is unitary) has a restriction which is a Cayley transform $V_K$, and this is so, by the extension theory of the Cayley transform, if and only if there is such a restriction of deficiency index $(1 : 1)$.

$U_H$ can be so restricted if and only if

$$\exists (0 \neq) \xi \quad \text{such that} \quad (1 - U_H)[\xi]^{\perp} \text{ is dense,}$$

and then $V_K = U_H$ on $[\xi]^{\perp}$, $= 0$ on $[\xi]$. (For $\mathcal{S} \subset \mathcal{H}$, $[\mathcal{S}]$ denotes the closed vector subspace generated by $\mathcal{S}$.)

There is no such restriction if and only if

$$\begin{aligned} \forall (0 \neq) \xi,\ \exists (0 \neq) \omega \quad \text{such that} \quad & \{(1 - U_H)[\xi]^{\perp}, \omega\} = 0, \\ & \Leftrightarrow \{[\xi]^{\perp}, (1 - U_H^*)\omega\} = 0, \\ & \Leftrightarrow (1 - U_H^*)\omega = x\xi \text{ for some } x. \qquad (*) \end{aligned}$$

Now it is immediate from the definition of the Cayley transform that 1 does not belong to its point spectrum nor to that of its adjoint, and so $(*)$ can be written

Received by the editors November 20, 1978.
*AMS* (*MOS*) *subject classifications* (1970). Primary 47B25.
*Key words and phrases*. Restriction of operator, Cayley transform, Hellinger-Toeplitz theorem.

$$(1 - U_H^*)\omega = \xi.$$

Since $U_H U_H^* = 1$ we may write this as

$$(1 - U_H)U_H^*\mathcal{H} = \mathcal{H}$$

Or, since $U_H$ is unitary,

$$(1 - U_H)\mathcal{H} = \mathcal{H}.$$

But $(1 - U_H)\mathcal{H} = \mathfrak{D}_H$, the domain of $H$, and now a generalization of the Hellinger-Toeplitz theorem [1, §25] shows that $H$ must be bounded.

## References

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