PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 76, Number 2, September 1979

TOTALLY GEODESIC FOLIATIONS ON 3-MANIFOLDS

DAVID L. JOHNSON AND LEE B. WHITT

ABSTRACT. If M is a compact 3-manifold, it is known that M can be foliated by 2-manifolds. Topological obstructions are given to the geodesibility of such a foliation $\mathfrak{F}$; that is, to the existence of a Riemannian metric on M making each leaf a totally geodesic submanifold. For example, $\pi_1(M)$ must be infinite, and hence the Reeb foliation of S^3 is not geodesible.

The study of foliations from the point of view of differential topology has made tremendous progress in recent years; the excellent survey article by Lawson [3] describes many of the most fruitful areas of research. Foliations are also of fundamental importance in differential geometry, particularly in the study of fiber bundles and connections, but the geometric aspects of foliations *per se* have received considerably less attention. This note considers a geometrization of these topological structures, namely totally geodesic foliations. That is, all leaves are required to be totally geodesic submanifolds. Such foliations arise naturally in Riemannian submersions [5]; also, a flat connection on a principal bundle yields a totally geodesic foliation for a suitable metric on the total space [2].

Two basic questions in this realm are:

Q1: Given a Riemannian manifold M, does it admit a totally geodesic foliation of a given codimension?

Q2: Given a foliation $\mathfrak{F}$ on a manifold M, is there a Riemannian metric on M such that $\mathfrak{F}$ is totally geodesic; namely is $\mathfrak{F}$ *geodesible*?

If the dimension of $\mathfrak{F}$ is one, H. Gluck has recently made significant inroads into these questions [1]. In particular, Gluck has shown that any closed orientable 3-manifold has a geodesic flow, and has characterized those flows on 2-manifolds that are geodesible. However, we show below that the codimension-one case is considerably more restrictive.

THEOREM 1. *If M is a compact 3-manifold that admits a codimension-one totally geodesic foliation for some Riemannian metric on M, then $\pi_1(M)$ is infinite.*

As a particularly interesting special case:

COROLLARY. *The Reeb foliation of S^3 is not geodesible.*

Received by the editors October 9, 1978 and, in revised form, December 30, 1978.

AMS (MOS) subject classifications (1970). Primary 57D30; Secondary 53B25, 53C20.

Key words and phrases. Foliations, geodesics, totally geodesic foliations, geodesibility, Reeb component.

0002-9939/79/0000-0435/$01.75

THEOREM 2. *Let $\mathcal{F}$ be a codimension-one foliation on a compact 3-manifold M. If either*

(i) *there is a closed, null-homotopic curve transverse to $\mathcal{F}$,*

(ii) *there is a leaf $L \in \mathcal{F}$ such that the induced map $\pi_1(L) \to \pi_1(M)$ is not injective,*

then $\mathcal{F}$ is not geodesible.

As a generalization of the methods used here, in a forthcoming paper the authors will show that, for M^n compact and $\mathcal{F}$ a codimension-one foliation with a compact leaf L_0, if $\mathcal{F}$ is geodesible, then, except for a possible Z_2 action, M is a fiber bundle over S^1, with fiber L_0. The foliation $\mathcal{F}$ need not, however, be the trivial one.

PROOF OF THEOREMS 1 AND 2. Let M^3 be a closed 3-manifold, and let $\mathcal{F}$ be a codimension-one foliation. By passing to a 2-fold cover we may assume $\mathcal{F}$ is transversally oriented. Assume that either

(i) $\pi_1(M)$ is finite,

(ii) there is a null-homotopic closed transversal, or

(iii) there is a leaf L with $\pi_1(L) \to \pi_1(M)$ not injective.

Note that these conditions would continue to hold on the cover if $\mathcal{F}$ were not transversally orientable.

The proof of these theorems relies heavily on a deep result of Novikov which states that if any of the above conditions holds, there is a compact leaf L_0; in fact there is a Reeb component bounded by L_0 [**4**].

Assume now that $\mathcal{F}$ is geodesible, and let a metric be chosen so that $\mathcal{F}$ is totally geodesic. If $\mathcal{H} = \mathcal{F}^{\perp}$, the orientability assumption implies there is a unit vector field X generating $\mathcal{H}$; in fact there are exactly two such fields. Choose X so that $X|_{L_0}$ is inward-pointing; that is, X points into the Reeb component bounded by L_0. Let γ be an integral curve of $\mathcal{H}$ with $\gamma(0) \in L_0$. As X is inward-pointing, $\gamma((0, \infty)) \subseteq \mathrm{Int}(L_0)$, the interior of the Reeb component, which is well-defined. In particular, γ is not closed.

It is evident that $\overline{\gamma((0, \infty))} - \gamma((0, \infty)) = \bar{\gamma} - \gamma$ is a union of integral curves of $\mathcal{H}$. Furthermore, $\mathrm{dist}(\bar{\gamma} - \gamma, L_0) = l$ is positive. Choose $x \in (\overline{\bar{\gamma} - \gamma})$, $y \in L_0$ realizing this distance, and let $\alpha(s)$ be a minimal geodesic from x to y, parametrized by arclength.

$\alpha'(l)$ is perpendicular to L_0 by the minimality of α, and also $\alpha'(0)$ is perpendicular to $\mathcal{H}_x$ by the same reasoning. However, as $\mathcal{F}$ is totally geodesic, and α at $s = 0$ is tangent to $\mathcal{F}$, α must be contained in some leaf of $\mathcal{F}$. But this contradicts the fact that α is perpendicular to $\mathcal{F}$ at y. Thus $\mathcal{F}$ cannot be geodesible.

REMARK. It is clear from the above argument that, if a codimension-one foliation $\mathcal{F}$ on a manifold M^n has a Reeb component, then it is not geodesible. In this setting a *Reeb component* consists of a leaf L_0 diffeomorphic to $S^1 \times S^{n-2}$ with $M - L_0$ disconnected, one component, the *interior* of

L_0, foliated by "snakes" diffeomorphic to $\mathbf{R}^{n-1}$. By passing to a two-fold cover if need be, it may be assumed that $\mathcal{F}$ is transversely orientable. The construction used in the proof of Theorems 1 and 2 then shows that $\mathcal{F}$ cannot be geodesible.

If M^n has zero Euler characteristic, Thurston [6] shows that there is a codimension-one foliation. Any such foliation has a closed transversal arc. It is not difficult to modify a foliation along such a closed transversal to introduce a Reeb component, using the tubular neighborhood theorem. From this follows

THEOREM 3. *Any compact manifold M^n with $\chi(M) = 0$ admits a codimension-one foliation that is not geodesible.*

REFERENCES

1. H. Gluck, *Open letter on geodesible flows*.

2. D. L. Johnson, *Kähler submersions and holomorphic connections* (to appear).

3. H. B. Lawson, *Foliations*, Bull. Amer. Math. Soc. **80** (1974), 369–418.

4. S. P. Novikov, *Topology of foliations*, Trudy Moskov. Mat. Obšč. **14** (1965), 248–278 = Trans. Moscow Math. Soc. **14** (1965), 268–304.

5. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.

6. W. P. Thurston, *Existence of codimension-one foliations*, Ann. of Math. (2) **104** (1976), 249–268.

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843