TENSOR PRODUCTS OF DIVISION RINGS AND FINITE GENERATION OF SUBFIELDS

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ABSTRACT. Let D be a division algebra over a field k . It is shown that if $D \otimes_k D^0$ is Noetherian, then every commutative subfield of D containing k is finitely generated over k . This theorem applies to D_n , the quotient division algebra of the n th Weyl algebra, and also to a number of other standard examples of nonalgebraic division algebras.

If the division ring D is finite dimensional over its center k , the (commutative) subfields of D play a fundamental role in the structure of D . By contrast, if D is not finite dimensional over k , very little is known about the subfields of D . The type of pathology that can arise is illustrated by examples of M. Smith [7, pp. 162–163]: She showed that there is a division algebra D with center k , containing two maximal subfields whose respective transcendence degrees are any two prescribed cardinal numbers.

For many of the familiar nonalgebraic division algebras D , insight into the subfields L of D can be obtained by use of the tensor product. In [4] it was shown that dimension properties of the rings $D \otimes_k k(x_1, \dots, x_n)$ can give an upper bound on the transcendence degree of L over k . Here we consider $D \otimes_k D^0$, where D^0 denotes the opposite algebra of D . Our basic result, Theorem 3, has a deceptively easy proof. It is worth noting that the consequences on finite generation of subfields given in Theorems 4 and 5 are new, and are results which seem completely inaccessible by direct computation.

Throughout this paper, k denotes a field and tensor products, when not specifically marked, are taken relative to k . The term “Noetherian” means left Noetherian and right Noetherian.

The approach to proving Theorem 3 is faithfully flat descent from a tensor product of division rings to a tensor product of fields. The applicability of this technique to problems in division algebras was first noted in [5]. We need a result on fields which is isolated as Proposition 1. A proof of this was recently given by Vámos.

PROPOSITION 1. *Let L be an extension field of k . Then $L \otimes L$ is Noetherian if and only if L is finitely generated over k .*

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PROOF. See [8]. $\square$

Before proceeding to the proof of Theorem 3, we make a direct application of Proposition 1 to a problem in rings with polynomial identity. It was proven in [6] that if the prime Noetherian PI-ring R is finitely generated over a central subfield k , then $R \otimes R^0$ is Noetherian. We give an example here which shows that this fails without the assumption of finite generation.

EXAMPLE 2. Let k be an arbitrary field, $A = k[y_1, y_2, \dots]$ a polynomial ring over k in countably many indeterminates, and let K be the field of fractions of A . Let Σ denote the permutation group of the set of positive integers and let Σ act on A and on K by permuting the indeterminates. Let $\alpha, \beta \in \Sigma$ be the transpositions

$$\alpha = (1, 2)(3, 4)(5, 6) \dots, \quad \beta = (2, 3)(4, 5)(6, 7) \dots,$$

and put $\gamma = \beta\alpha$. While β and α are each of order two, γ is the infinite cycle

$$\gamma = (\dots, 8, 6, 4, 2, 1, 3, 5, 7, \dots).$$

If $K_1 = K^\alpha$ is the fixed field of α and $K_2 = K^\beta$ is that of β , then $K_1 \cap K_2 \subset K^\gamma = k$. The last equality follows easily from the “lowest terms” representation, unique up to units, of an element of the quotient field of a unique factorization domain. We thus obtain a tower of fields

where K has dimension two over K_1 and over K_2 , but is not even finitely generated over the intersection.

Now let $K[x]$ be the ring of polynomials over K and let $K_i + xK[x]$, $i = 1, 2$, denote the subring of $K[x]$ consisting of those polynomials whose constant term lies in K_i . In the ring of 2×2 matrices over $K[x]$ let R be defined by

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a \in K_1 + xK[x], \\ d \in K_2 + xK[x], \\ b, c \in xK[x] \end{array} \right\}$$

As is easily verified R is a prime Noetherian PI-ring with center $C \simeq k + xK[x]$. There is a surjective C -algebra homomorphism $R \rightarrow K_1$, obtained by mapping each matrix to its upper left entry, then reducing mod $xK[x]$. This map induces a ring epimorphism $R \otimes_C R^0 \rightarrow K_1 \otimes_C K_1 \simeq K_1 \otimes_k K_1$. Proposition 1 says that the image is not Noetherian; hence $R \otimes_C R^0$ cannot be Noetherian.

We note in passing that $R \otimes_C R^0$ is a semiprime ring, but is not prime.

THEOREM 3. *Let D be a division algebra over k , $L \supset k$ a commutative subfield of D :*

- (i) *$D \otimes L$ is Noetherian if and only if L is a finitely generated extension of k ;*
- (ii) *If $D \otimes D^0$ is Noetherian then L is finitely generated over k .*

PROOF. (i) Suppose $D \otimes L$ is Noetherian. Since D is a (left, say) vector space over L , $D \otimes L$ is a free $(L \otimes L)$ -module. This implies that $D \otimes L$ is a left faithfully flat extension of $L \otimes L$ [2, pp. 50–52], and that $I(D \otimes L) \cap L \otimes L = I$ for any ideal I of $L \otimes L$. Thus $L \otimes L$ is Noetherian and, by Proposition 1, L is finitely generated. Conversely, if L is finitely generated and $\{t_1, \dots, t_n\}$ is a transcendence base for L over k , then $D \otimes k(t_1, \dots, t_n)$ is isomorphic to a localization of a polynomial ring over D and is Noetherian. As $D \otimes L$ is a finite module over this ring, it too is Noetherian.

(ii) If $D \otimes D^0$ is Noetherian then by faithful flatness $D \otimes L$ is Noetherian. Now apply (i). $\square$

We first apply Theorem 3 to the quotient division ring of the Weyl algebra.

THEOREM 4. *Let k be a field of characteristic zero, $A_n = A_n(k)$ the Weyl algebra over k and D_n the quotient division ring of A_n . Then*

- (i) *every maximal subfield of D_n is finitely generated over k .*
- (ii) *k is algebraically closed in D_n ; that is, if $\alpha \in D_n$ is algebraic over k then $\alpha \in k$.*

PROOF. (i) We first recall the definition: A_n is the k -algebra with $2n$ generators $p_1, \dots, p_n; q_1, \dots, q_n$ and relations $[p_i, q_j] = p_i q_j - q_j p_i = \delta_{ij}$ (Kronecker delta), $[p_i, p_j] = 0 = [q_i, q_j]$. We further recall two properties of Weyl algebras: (1) for any k -algebra B , $A_n \otimes_k B \simeq A_n(B)$ where $A_n(B)$ is the n th Weyl algebra with coefficients from B ; (2) $A_n(B)$ is Noetherian if and only if B is Noetherian [3, p. 439]. In particular, if $L \subset D_n$ is a maximal subfield then $A_n \otimes L$ is Noetherian and so also is its partial quotient ring $D_n \otimes L$. Finite generation of L follows from Theorem 3(i).

(ii) Let $\alpha \in D_n - k$ and put $L = k(\alpha)$. As $D_n \otimes L$ is a localization of $A_n(L)$, it is a domain. Since $L \otimes L$ embeds in $D_n \otimes L$ it also is a domain, and this can hold only if α is transcendental over k [3, p. 248]. $\square$

It is also known [4] that every maximal subfield of D_n has transcendence degree at most n over k , but it is not known whether the value is always precisely n —this appears to be quite a difficult open question. In addition, it is known that D_n contains maximal subfields which are not purely transcendental over k . Dixmier has given an example [9, p. 222] of a subfield of D_1 not purely transcendental over k , and a lemma of Roquette [10, p. 209] shows that this field cannot be embedded in any purely transcendental extension of k .

Let L be a field of characteristic zero and let d be a nonzero derivation of L with $\text{Ker } d = k$. An Ore domain related to A_1 is the ring of differential operators over L , $R = L[y; d]$ [3, 12.2]. In [1], Amitsur proved that the

centralizer of any $g \in R - L$ is commutative and is a finite free module over $k[g]$. Hence, every maximal commutative subalgebra of R not contained in L is finitely generated over k . Let D be the quotient division ring of R . It is reasonable to expect an analogous description of the maximal subfields of D , although such a result does not seem to be deducible from Amitsur's work on R . Instead, we apply Theorem 3.

THEOREM 5. *Let k, L, R, D be as in the preceding paragraph. Then the following are equivalent:*

- (i) $D \otimes D^0$ is Noetherian;
- (ii) every subfield of D containing k is finitely generated over k ;
- (iii) L is finitely generated over k .

PROOF. (i) $\Rightarrow$ (ii). Theorem 3.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Let $A = L \otimes D^0$ and extend d to A by putting $d(f \otimes \alpha) = f^d \otimes \alpha$. A is Noetherian by Theorem 3(i) and hence so is $A[y; d] \simeq L[y; d] \otimes D^0$. As $D \otimes D^0$ is a localization of this ring it too is Noetherian. $\square$

We remark that a result completely analogous to Theorem 5 holds for quotient rings of twisted polynomial rings of the form $L[x; \sigma]$ where L is a field extension of k and σ is a k -automorphism of L .

NOTE ADDED IN PROOF. A. Joseph has informed us that he has found an example of a maximal subfield of transcendence degree 1 in the Weyl division ring D_2 , thus settling the question raised after the proof of Theorem 4.

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