A THEOREM FOR PRIME RINGS

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ABSTRACT. Let n be a positive integer and let R be a prime ring either of characteristic zero or of characteristic $> n$. Then for any $a_1, a_2, \dots, a_{n+1} \in R$, if $a_1 x a_2 x \cdots a_n x a_{n+1} = 0$ for all $x \in R$. Then $a_i = 0$ for some $1 \leq i \leq n+1$.

In [1, Lemma 2] Posner showed that given any three elements a_1, a_2 , and a_3 in a prime ring R , if $a_1 x a_2 x a_3 = 0$ for all $x \in R$, then at least one of a_1, a_2, a_3 is zero. The purpose of this note is to prove the following generalization of Posner's result.

THEOREM. Let n be a positive integer and let R be a prime ring either of characteristic zero or of characteristic $\geq n$. Then for any $a_1, a_2, \dots, a_{n+1} \in R$, if $a_1 x a_2 x \cdots a_n x a_{n+1} = 0$ for all $x \in R$, then $a_i = 0$ for some $1 \leq i \leq n+1$.

It should be noted that the theorem may not be true if the restriction on the characteristic of R is removed. Let R be the prime ring of $m \times m$ matrices over a finite field. We wish to show that for any $x \in R$, there exists a nonzero $a_x \in R$ such that $(a_x x)^2 = 0$. First pick some nonzero $a \in R$ such that $a^2 = 0$. Then for $x \in R$, if x is a unit let $a_x = ax^{-1}$ and if x is not a unit, there exists a nonzero $a_x \in R$ such that $a_x x = 0$. In any case $a_x \neq 0$, and the equation

$$\prod_{x \in R, x \neq 0} (a_x y)^2 = 0$$

holds for all $y \in R$.

We begin by linearizing the identity $a_1 x a_2 x \cdots a_n x a_{n+1} = 0$.

PROPOSITION 1. Let $a_1, a_2, \dots, a_{n+1}$ be elements of a ring R . Suppose $a_1 x a_2 x \cdots a_n x a_{n+1} = 0$ for all $x \in R$. Then for all $x_1, x_2, \dots, x_n \in R$,

$$\sum_{\sigma \in S_n} a_1 x_{\sigma(1)} a_2 x_{\sigma(2)} \cdots a_n x_{\sigma(n)} a_{n+1} = 0$$

where S_n is the symmetric group of degree n .

PROOF. For $J \subseteq \{1, 2, \dots, n\}$, by $\sum_J a_1 x_{k_1} a_2 x_{k_2} \cdots a_n x_{k_n} a_{n+1}$ we mean the sum over all indices $k_1, k_2, \dots, k_n$ such that $\{k_1, k_2, \dots, k_n\} = J$. Observe that for any integer $1 \leq j \leq n$ and for any $x_1, x_2, \dots, x_j \in R$,

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$$\begin{aligned} \sum_{k_1, k_2, \dots, k_n=1}^j a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} \\ = \sum_{J \subseteq \{1, 2, \dots, j\}} \sum_J a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1}. \end{aligned}$$

Now by hypothesis, for any $x \in R$, $a_1 x a_2 x \dots a_n x a_{n+1} = 0$. Proceeding inductively, let $1 < j < n$ and suppose we have shown for all $1 < i < j$ and for all $x_1, x_2, \dots, x_i \in R$, that

$$\sum_{\{1, 2, \dots, i\}} a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} = 0.$$

Observe that this implies for any set $J \subseteq \{1, 2, \dots, n\}$ of cardinality $< j$ and any $\{x_i; i \in J\} \subseteq R$,

$$\sum_J a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} = 0.$$

For if $J = \{l_1, l_2, \dots, l_i\}$ where $l_1 < l_2 < \dots < l_i$ let $y_k = x_{l_k}$ for $k = 1, 2, \dots, i$. Then

$$\sum_J a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} = \sum_{\{1, 2, \dots, i\}} a_1 y_{k_1} a_2 y_{k_2} \dots a_n y_{k_n} a_{n+1} = 0.$$

Let $x_1, x_2, \dots, x_{j+1} \in R$ and let $x = x_1 + x_2 + \dots + x_{j+1}$. Then

$$\begin{aligned} 0 &= a_1 x a_2 x \dots a_n x a_{n+1} = \sum_{k_1, k_2, \dots, k_n=1}^{j+1} a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} \\ &= \sum_{J \subseteq \{1, 2, \dots, j+1\}} \sum_J a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} \\ &= \sum_{\{1, 2, \dots, j+1\}} a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} \\ &\quad + \sum_{J \subsetneq \{1, 2, \dots, j+1\}} \sum_J a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} \\ &= \sum_{\{1, 2, \dots, j+1\}} a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1}. \end{aligned}$$

This completes the inductive step. Thus

$$\begin{aligned} 0 &= \sum_{\{1, 2, \dots, n\}} a_1 x_{k_1} a_2 x_{k_2} \dots a_n x_{k_n} a_{n+1} \\ &= \sum_{\sigma \in S_n} a_1 x_{\sigma(1)} a_2 x_{\sigma(2)} \dots a_n x_{\sigma(n)} a_{n+1}. \end{aligned}$$

A proof of the following lemma may be found in [2, Lemma 1].

LEMMA 1. *Let R be a ring and suppose there exists a right ideal $A \neq (0)$ in R and an integer $n > 1$ such that for all $x \in A$, $x^n = 0$. Then R contains a nonzero nilpotent ideal.*

Since a prime ring can contain no nonzero nilpotent ideal, we have:

COROLLARY 1. *No prime ring can satisfy the hypothesis of Lemma 1.*

PROOF OF THE THEOREM. We proceed by induction on n . Suppose $n > 1$. We first use the induction hypothesis to establish:

PROPOSITION 2. *Let $a_1, a_2, \dots, a_{n+1} \in R$ such that $a_1 x a_2 x \dots a_n x a_{n+1} = 0$ for all $x \in R$. Suppose $a_i \neq 0$ for all $1 \leq i \leq n+1$. Then if for some $1 \leq i \leq n$ and some $r \in R$, $a_i r = 0$, then $a_i = 0$ for all $1 \leq i \leq n$.*

PROOF. Let

$$J = \{i: 1 \leq i \leq n \text{ and } a_i r = 0\} = \{k_1, k_2, \dots, k_j\}$$

where $k_1 < k_2 < \dots < k_j$. Suppose $j < n$. Let $y_1, y_2 \in R$. Let $x_i = y_1$ for $i \in J$ and $x_i = ry_2$ for $i \in \{1, 2, \dots, n\} - J$. Then for $\sigma \in S_n$, if $\sigma(J) = J$, then

$$a_1 x_{\sigma(1)} a_2 x_{\sigma(2)} \dots a_n x_{\sigma(n)} a_{n+1} = a_1 x_1 a_2 x_2 \dots a_n x_n a_{n+1}$$

and if $\sigma(J) \neq J$, then since, for some $i \in J$, $a_i x_{\sigma(i)} = a_i ry_2 = 0$, we have

$$a_1 x_{\sigma(1)} a_2 x_{\sigma(2)} \dots a_n x_{\sigma(n)} a_{n+1} = 0.$$

Thus, by Proposition 1,

$$\begin{aligned} 0 &= \sum_{\sigma \in S_n} a_1 x_{\sigma(1)} a_2 x_{\sigma(2)} \dots a_n x_{\sigma(n)} \\ &= \sum_{\sigma \in S_n, \sigma(J)=J} a_1 x_1 a_2 x_2 \dots a_n x_n a_{n+1} \\ &= j! (n-j)! a_1 x_1 a_2 x_2 \dots a_n x_n a_{n+1}. \end{aligned}$$

Now either $\text{char}(R) = 0$ or $\text{char}(R) \geq n$. If $\text{char}(R) \geq n$, then, since $\text{char}(R)$ is a prime integer and, by assumption, $1 \leq j < n$, $\text{char}(R) \nmid j!(n-j)!$. Thus in either case

$$0 = a_1 x_1 a_2 x_2 \dots a_n x_n a_{n+1}.$$

We now consider two cases.

Case 1. Suppose $k_1 = 1$. Let $k_{j+1} = n+1$. For $2 \leq i \leq j+1$, let

$$q_i = \begin{cases} a_{k_{i-1}+1} r y_2 a_{k_{i-1}+2} r y_2 \dots a_{k_{i-1}} r y_2 a_{k_i} & \text{if } k_{i-1} + 1 < k_i, \\ a_{k_i} & \text{if } k_{i-1} + 1 = k_i. \end{cases}$$

Note that $q_i = a_{k_{i-1}+1} x_{k_{i-1}+1} a_{k_{i-1}+2} x_{k_{i-1}+2} \dots a_{k_{i-1}} x_{k_{i-1}} a_{k_i}$ and since $k_1 = 1$,

$$\begin{aligned} 0 &= a_1 x_1 (a_{k_1+1} x_{k_1+1} \dots a_{k_2-1} x_{k_2-1} a_{k_2}) x_{k_2} (a_{k_2+1} x_{k_2+1} \dots a_{k_3-1} x_{k_3-1} a_{k_3}) x_{k_3} \\ &\quad \dots x_{k_j} (a_{k_j+1} x_{k_j+1} \dots a_n x_n a_{n+1}) \\ &= a_1 x_1 q_2 x_{k_2} \dots q_j x_{k_j} q_{j+1} = a_1 y_1 q_2 y_1 \dots q_j y_1 q_{j+1}. \end{aligned}$$

Since $j < n$ and this equation is valid for all $y_1 \in R$, we may use our induction hypothesis and the assumption $a_1 \neq 0$ to conclude $q_i = 0$ for some

$2 \leq i \leq j + 1$. But then

$$0 = q_2 y_2 q_3 y_2 \cdots q_j y_2 q_{j+1} = t_2 y_2 t_3 y_2 \cdots t_n y_2 a_{n+1}$$

where for $2 \leq i \leq n$,

$$t_i = \begin{cases} a_i & \text{if } i \in J, \\ a_i r & \text{if } i \notin J. \end{cases}$$

Again using the induction hypothesis and the assumption that $a_i \neq 0$ for all $2 \leq i \leq n + 1$, we concluded that for some $i \notin J$, $a_i r = 0$ which contradicts the very definition of J .

Case 2. Suppose $k_1 \neq 1$. Let

$$I = \{1, 2, \dots, n\} - J = \{i_1, i_2, \dots, i_{n-j}\}$$

where $i_1 < i_2 < \cdots < i_{n-j}$. Then $i_1 = 1$ and $a_1 r \neq 0$. Let $i_{n-j+1} = n + 1$. For $2 \leq k \leq n - j + 1$, let

$$q_k = \begin{cases} a_{i_{k-1}+1} y_1 a_{i_{k-1}+2} y_1 \cdots a_{i_k-1} y_1 a_{i_k} r & \text{for } i_{k-1} + 1 < i_k, \\ a_{i_k} r & \text{for } i_{k-1} + 1 = i_k. \end{cases}$$

We then proceed in a manner similar to that in Case 1. That is, we have

$$0 = a_1 r y_2 q_2 y_2 \cdots q_{n-j} y_2 q_{n-j+1}.$$

Then, by the induction hypothesis and since $a_1 r \neq 0$, $q_i = 0$ for some $2 \leq i \leq n - j + 1$. Thus,

$$0 = q_2 y_1 q_3 y_1 \cdots q_{n-j} y_1 q_{n-j+1} = t_2 y_1 t_3 y_1 \cdots t_n y_1 a_{n+1}$$

where, for $2 \leq i \leq n$,

$$t_i = \begin{cases} a_i & \text{if } i \in J, \\ a_i r & \text{if } i \in I. \end{cases}$$

As before we reach the contradiction $a_i r = 0$ for some $i \notin J$. Thus $j = n$ and we are done.

PROPOSITION 3. Let $a_1, a_2, \dots, a_{n+1} \in R$ such that $a_1 x a_2 x \cdots a_n x a_{n+1} = 0$ for all $x \in R$. Suppose $a_i \neq 0$ for all $1 \leq i \leq n + 1$. Then if for some $2 \leq i \leq n + 1$, $ra_i = 0$, then $ra_i = 0$ for all $2 \leq i \leq n + 1$.

The proof of Proposition 3 is similar to that of Proposition 2 and is omitted.

PROPOSITION 4. Let $a_1, a_2, \dots, a_{n+1} \in R$ such that for all $x \in R$,

$$a_1 x a_2 x \cdots a_n x a_{n+1} = 0.$$

Suppose $a_i \neq 0$ for all $1 \leq i \leq n + 1$. Suppose further that for some $2 \leq j \leq n$, $a_1 = a_2 = \cdots = a_j$ and for some $r, s \in R$, $ra_1 s = 0$. Then $ra_i s = 0$ for all $1 \leq i \leq n$.

PROOF. If $ra_1 = 0$, then $ra_2 = ra_1 = 0$ and so by Proposition 3, $ra_i = 0$ for all $2 \leq i \leq n + 1$. If $ra_1 \neq 0$, then by Proposition 2 and since $(ra_1)s = 0$, $a_i s = 0$ for all $2 \leq i \leq n$. In either case $ra_i s = 0$ for all $1 \leq i \leq n$.

PROPOSITION 5. Let $2 \leq j < n$ and let $a_j, a_{j+1}, \dots, a_{n+1} \in R$ such that

$$(a_j x)^j a_{j+1} x \dots a_n x a_{n+1} = 0$$

for all $x \in R$. Suppose $a_i \neq 0$ for all $j \leq i \leq n+1$. Then

$$(a_{j+1} x)^{j+1} a_{j+2} x \dots a_n x a_{n+1} = 0$$

for all $x \in R$.

PROOF. Let $r = (a_j x)^{j-1}$, $s = x a_{j+1} x \dots a_n x a_{n+1}$. Then $r a_j s = 0$ and so by Proposition 4, $r a_i s = 0$ for all $j \leq i \leq n$. In particular

$$0 = r a_{j+1} s = (a_j x)^{j-1} (a_{j+1} x)^2 a_{j+2} x \dots a_n x a_{n+1}.$$

If $j-1 \geq 2$, let

$$r = (a_j x)^{j-2}, \quad s = x (a_{j+1} x)^2 a_{j+2} x \dots a_n x a_{n+1}.$$

Then $r a_j s = 0$ and again by Proposition 4,

$$0 = r a_{j+1} s = (a_j x)^{j-2} (a_{j+1} x)^3 a_{j+2} x \dots a_n x a_{n+1}.$$

Continuing in this way we eventually reach

$$0 = a_j x (a_{j+1} x)^j a_{j+2} x \dots a_n x a_{n+1}.$$

Then let $r = x (a_{j+1} x)^j a_{j+2} x \dots a_n x a_{n+1}$. Then $a_j r = 0$ and so by Proposition 2,

$$0 = a_{j+1} r = (a_{j+1} x)^{j+1} a_{j+2} x \dots a_n x a_{n+1}.$$

It is now quite easy to conclude the proof of the Theorem. Let $a_1, \dots, a_{n+1} \in R$ such that $a_1 x a_2 x \dots a_n x a_{n+1} = 0$ for all $x \in R$. Suppose $a_i \neq 0$ for all $1 \leq i \leq n+1$. Let $r = x a_2 x \dots a_n x a_{n+1}$. Then $a_1 r = 0$ and so by Proposition 2,

$$0 = a_2 r = (a_2 x)^2 a_3 x \dots a_n x a_{n+1}.$$

Then repeated applications of Proposition 5 eventually yield $(a_n x)^n a_{n+1} = 0$. Setting $r = (a_n x)^n$ in Proposition 3 yields $(a_n x)^n a_n = 0$. But then $a_n R$ is a right ideal in R and for all $x \in a_n R$, $x^{n+1} = 0$. By Corollary 1, $a_n R = (0)$. But then since R is a prime ring, $a_n = 0$.

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