D'ALEMBERT'S FUNCTIONAL EQUATION ON GROUPS

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ABSTRACT. Group representation theory is used to solve a functional equation of the D'Alembert type on groups.

I. Reduction to a representation theory problem. We consider a functional equation of the form

$$\phi_1(gh) + \phi_2(g^{-1}h) = \sum_{i=1}^n \kappa_i(g)\lambda_i(h) \quad (*)$$

where $\phi_1, \phi_2, \kappa_i, \lambda_i, i = 1, \dots, n$, are some (measurable) complex functions given on a locally compact group G . We can and shall assume that functions κ_i and $\lambda_i, i = 1, \dots, n$, are linearly independent. The equation $(*)$ can be viewed as a generalization of the known D'Alembert functional equation

$$\phi(gh) + \phi(g^{-1}h) = 2\phi(g)\phi(h),$$

which has been studied by many authors (cf. [4]–[6]). A particular case of $(*)$ when $\phi_1 = -\phi_2$ arises in some statistical applications and was studied by one of the authors in the situation when G is a compact Lie group [7]. The solution of $(*)$ in the case $G = R^1$ can be found in Aczél's book [1, pp. 171–176, 199].

In this paper we present the solution of $(*)$ in the case when

$$\int |\phi_i(g)|^2 d\nu(g) < \infty, \quad i = 1, 2,$$

where ν is the left Haar measure on G . Henceforth we denote the space of such functions by $L_2(G)$. We also consider the case when $\phi_i, i = 1, 2$, can be represented as a finite combination of positive definite functions.

Note that if $\varphi(g) = \phi_1(g) + \phi_2(g)$, $\xi(g) = \phi_1(g) - \phi_2(g)$ then

$$\begin{aligned} \varphi(gh) + \varphi(g^{-1}h) &= \sum_{i=1}^n [\kappa_i(g) + \kappa_i(g^{-1})]\lambda_i(h) \\ &= \sum_{i=1}^m \alpha_i(g)\varphi_i(h) \end{aligned} \quad (**)$$

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and

$$\begin{aligned}\xi(gh) - \xi(g^{-1}h) &= \sum_{j=1}^n [\kappa_i(g) - \kappa_j(g^{-1})] \lambda_j(h) \\ &= \sum_{j=1}^r \beta_j(g) \xi_j(h)\end{aligned}\quad (***)$$

with linearly independent functions $\alpha_i, \varphi_i, i = 1, \dots, m$, and $\beta_j, \xi_j, j = 1, \dots, r$. Therefore we restrict our attention to the case $\phi_1 = \phi_2$ or $\phi_1 = -\phi_2$ in (*).

We reduce the solution of (*) to a certain problem in representation theory in the following way. Let us begin with the equation (**). If H denotes the closure in $L_2(G)$ of the linear space spanned by the left shifts $\varphi(g \cdot), g \in G$, of the function φ then the left regular representation U acts in H : $U(g)\eta(\cdot) = \eta(g^{-1} \cdot), \eta \in H$, and $U(g)$ is a unitary operator. The relation (**) implies that

$$[U(g) + U(g^{-1})]\varphi = \sum_{i=1}^m \alpha_i(g) \varphi_i,$$

where φ denotes the vector of L corresponding to the function $\varphi(\cdot)$, and $\varphi_1, \dots, \varphi_m$ are vectors from H . Because of the definition of H , U is a cyclic representation with a cyclic vector φ (i.e., the space spanned by vectors $U(g)\varphi, g \in G$, is dense in H). The following result is a corollary of these considerations and Theorem 2.1.

THEOREM 1.1. *Let G be a locally compact group of type one such that the elements of the form $g^4, g \in G$, generate a dense subgroup of G . Every solution $\varphi \in L_2(G)$ of the equation (**) has the form*

$$\varphi(g) = \langle U(g^{-1})\varphi, \eta \rangle$$

where U is a finite dimensional, unitary, cyclic (with a cyclic vector φ) representation of G such that the space spanned by vectors $[U(g) + U(g^{-1})]\varphi, g \in G$ has dimension m , and η is some vector of the representation space U .

Note that a formula for the dimension of the representation U can be obtained from Theorem 2.1.

It is immediately seen that the functional equation (***) is equivalent to the finite dimensionality of the space spanned by vectors $[U(g) - U(g^{-1})]\xi, g \in G$, where again U is a unitary representation with a cyclic vector ξ .

THEOREM 1.2. *Let G be a locally compact group of type one such that the elements of the form $g^2, g \in G$, generate a dense subgroup of G . Every solution $\xi \in L_2(G)$ has the same form as indicated in Theorem 1.1 with ξ instead of φ , $[U(g) - U(g^{-1})]\xi$ instead of $[U(g) + U(g^{-1})]\varphi$ and r instead of m .*

The solution of the general equation (*) now follows easily from Theorems 1.1 and 1.2.

The content of these theorems is that if G is noncompact then the D'Alembert functional equation has few solutions, as noncompact groups usually have few finite dimensional unitary representations.

THEOREM 1.3. *Under the assumptions of Theorem 1.1, if there exists a nonzero solution of (**) then G is compact.*

PROOF. It follows from Theorem 1.1 that every solution of (**) has the form

$$\varphi(g) = \langle U(g^{-1})\varphi, \eta \rangle$$

with a finite dimensional representation U . However, such a matrix element cannot be square integrable unless G is compact. Indeed let K be the kernel of U . Clearly K is a closed, normal subgroup of G and φ is constant on cosets of K in G . In order for φ to be square integrable, K must have finite volume with respect to Haar measure which implies the compactness of K .

To prove that G is compact, it suffices to show that G/K is compact. To this end we may assume that $K = \{e\}$ so that U is injective. But then G is compactly injectible, and hence C is the product of a compact group and R^p for some p (cf. [2, §16.4.2]). But R^p has no injective, finite dimensional unitary representations unless $p = 0$. Thus G is compact.

We give another version of Theorem 1.1.

THEOREM 1.4. *Theorem 1.1 holds if one assumes that φ is a linear combination of positive definite functions instead of $\varphi \in L_2(G)$.*

PROOF. It follows from Godement [3] that there exists a unitary representation U and vectors φ and η such that

$$\varphi(g) = \langle U(g^{-1})\varphi, \eta \rangle.$$

These vectors φ and η can be assumed to be cyclic for U (the latter since $\varphi(g) = \langle \varphi, U(g)\eta \rangle$). Moreover we can replace η by its projection onto the closed subspace spanned by the vectors $U(g^{-1})\varphi$. Because of (**) the space of functions $\langle [U(g) + U(g^{-1})]\varphi, U(\cdot)\eta \rangle$, $g \in G$, is finite dimensional. The cyclicity of η implies that the space spanned by the vectors $[U(g) + U(g^{-1})]\varphi$ is finite dimensional and the cyclicity of φ and Theorem 2.1 imply that U is finite dimensional.

2. Symmetric and antisymmetric intertwining operators. Let U be a unitary representation of the topological group G in a Hilbert space H . Let H^* be the continuous dual of H and let U^* be the contragradient representation to U , i.e.,

$$\langle \xi, U^*(g)\eta \rangle = \langle U(g^{-1})\xi, \eta \rangle$$

Clearly H^* is a Hilbert space which is conjugate isomorphic to H and U^* is a unitary operator.

Let $I(U, U^*)$ denote the set of all continuous operators A mapping H into

U^* such that

$$U^*(g)A = AU(g).$$

If A^* is the dual operator, $A^*: H^{**} = H \rightarrow H^*$, then A is said to be symmetric if $A^* = A$ and antisymmetric if $A^* = -A$. The space of symmetric elements of $I(U, U^*)$ is denoted $I_s(U, U^*)$ and the space of antisymmetric elements is denoted $I_a(U, U^*)$. Clearly

$$I(U, U^*) = I_s(U, U^*) + I_a(U, U^*).$$

Note that A^* is not the same as the Hilbert space adjoint of A which is a mapping of H^* to H .

We prove the following.

THEOREM 2.1. *Let G be a locally compact group and U a type one unitary representation of G which possesses a cyclic vector φ . Let L_+ and L_- be the subspaces of H defined as*

$$L_{\pm} = \text{span}\{[U(g) \pm U(g^{-1})]\varphi, g \in G\}.$$

Then:

(a) *if L_- is finite dimensional and the elements of the form $g^2, g \in G$, generate a dense subgroup of G , then U is finite dimensional and*

$$\dim U = \dim L_- + \dim I_s(U, U^*);$$

(b) *if L_+ is finite dimensional and the elements of the form $g^4, g \in G$, generate a dense subgroup of G then U is finite dimensional and*

$$\dim U = \dim L_+ + \dim I_a(U, U^*).$$

PROOF. Since the proofs of (a) and (b) are similar we prove only (b). Let L' be the annihilator of L_+ in H^* . We shall establish a one-to-one correspondence between L' and $I_a(U, U^*)$. Specifically, the correspondence will be obtained as follows. Let $\lambda \in L'$. For each vector of the form

$$\xi = \sum_i c_i U(g_i)\varphi$$

define

$$B_{\lambda}(\xi) = \sum_i c_i U^*(g_i)\lambda.$$

We shall show that this correspondence is well defined. Granting this, B_{λ} becomes a densely defined intertwining operator from H to H^* . The main problem in proving our theorem is to demonstrate that B_{λ} is, in fact, a bounded operator. This will be achieved by expanding the domain of B as much as possible.

Now, let $V = U \oplus U^*$, and let $\mathcal{Q}$ be the von Neumann algebra on $H \oplus H^*$ generated by the set of operators $\{V(g), g \in G\}$. Each element of $\mathcal{Q}$ is an operator of the form $A \oplus A'$ where

$$A = \lim_{\alpha} \sum_{g \in G} c^{\alpha}(g)U(g), \quad A' = \lim_{\alpha} \sum_{g \in G} c^{\alpha}(g)U^*(g),$$

and c^α is a net of functions on G which are supported on finite sets on G . The limits are taken in the strong operator topology.

LEMMA 2.1. *Suppose $A \oplus A' \in \mathcal{Q}$ and $\lambda \in L'$. Then $A\varphi = 0$ implies that $A'\lambda = 0$.*

PROOF. Let A and A' be represented as described above and consider the function

$$f(h) = \langle U(h)\varphi, A'\lambda \rangle.$$

By definition

$$f(h) = \lim_\alpha \sum_{g \in G} \langle U(h)\varphi, c^\alpha(g)U^*(g)\lambda \rangle = \lim_\alpha \sum_{g \in G} \langle U(g^{-1}h)\varphi, \lambda \rangle c^\alpha(g).$$

Since $\lambda \in L'$,

$$\langle U(g)\varphi, \lambda \rangle = -\langle U(g^{-1})\varphi, \lambda \rangle.$$

Thus

$$\begin{aligned} f(h) &= \lim_\alpha \sum_{g \in G} \langle U(h^{-1})c_\alpha(g)U(g)\varphi, \lambda \rangle \\ &= \langle U(h^{-1})A\varphi, \lambda \rangle = 0 \end{aligned}$$

and $A'\lambda = 0$ because of the cyclicity of φ . The lemma is proven.

Let $\mathcal{C}$ be the subspace of H defined by $\mathcal{C} = \{A\varphi : A \oplus A' \in \mathcal{Q} \text{ for some } A'\}$. For each $\lambda \in L'$ let B_λ be the mapping of $\mathcal{C}$ into H^* defined as $B_\lambda(A\varphi) = A'\lambda$ where $A \oplus A' \in \mathcal{Q}$. By Lemma 2.1 this definition makes sense.

LEMMA 2.2. *For all $v, w \in \mathcal{C}$,*

(i) $\langle v, B_\lambda w \rangle = -\langle w, B_\lambda v \rangle$, *and*

(ii) $B_\lambda U(g)v = U^*(g)B_\lambda v$.

Conversely, any linear operator $B: \mathcal{C} \rightarrow H^$ which satisfies (i) and (ii) has the form $B = B_\lambda$ where $\lambda \in L'$.*

PROOF. The proof of (i) is analogous to that of Lemma 2.1 and (ii) follows from the definition of B_λ . The last statement of Lemma 2.2 is true since

$$\begin{aligned} \langle U(g)\varphi, B\varphi \rangle &= -\langle \varphi, BU(g)\varphi \rangle = -\langle \varphi, U^*(g)B\varphi \rangle \\ &= \langle -U(g^{-1})\varphi, B\varphi \rangle \end{aligned}$$

Now let $\pi: H \rightarrow H$ be a central projection for U , i.e. π commutes with U and with the commuting algebra of U . We call π balanced if $\pi \oplus \pi^*$ is a central projection for V . More specifically, the general "matrix" form of an intertwining operator for V is $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha \in I(U, U)$, $\delta \in I(U^*, U^*)$, $\beta \in I(U^*, U)$ and $\gamma \in I(U, U^*)$. Hence π is balanced if π is central for U and $\gamma\pi = \pi^*\gamma$ for all $\gamma \in I(U, U^*)$. (The β -identity follows from the γ -identity by transposition.) Intuitively, "balanced central projection" means that if an irreducible representation U_0 "occurs" in π , then U_0^* also "occurs" in π .

given that U_0^* "occurs" in U . Of course, U might have no discrete spectrum so this is only formal.

LEMMA 2.3. *If π is balanced, then $\pi L_+ \subset L_+$. If there is a nonzero balanced projection π such that $\pi L_+ = 0$ then the image of π is one-dimensional and U is trivial on the image of π .*

PROOF. It suffices to show that π^* maps L' into L' . Let $\lambda \in L'$. Then $\lambda = B_\lambda \varphi$ and $\pi^* \lambda = \pi^* B_\lambda \varphi$. But then

$$\begin{aligned} \langle U(g)\varphi, \pi^* \lambda \rangle &= \langle U(g)\varphi, \pi^* B_\lambda \varphi \rangle = -\langle \varphi, B_\lambda \pi U(g)\varphi \rangle \\ &= -\langle \varphi, \pi^* U^*(g)\lambda \rangle = -\langle U(g^{-1})\varphi, \pi^* \lambda \rangle \end{aligned}$$

since $\pi U(g) \oplus \pi^* U^*(g) \in \mathcal{Q}$. Thus, as claimed, $\pi^* \lambda \in L'$.

Now, if $\pi(U(g) + U(g^{-1}))\varphi = 0$ then $U(g)\pi\varphi = -U(g^{-1})\pi\varphi$, so that $U(g^4)\pi\varphi = \pi\varphi$. Since the elements g^4 generate a dense subgroup, U is trivial on the image of π . Since a cyclic representation can contain the identity representation at most once, our lemma follows.

COROLLARY. *There is only a finite number of disjoint balanced projections for U . Also $\sum_1^q \pi_i = 1$ if $\pi_1, \dots, \pi_q$ is a maximal family of disjoint balanced projections.*

PROOF. Let π_1 be the unique balanced projection such that $\pi_1 L_+ = 0$ (it is possible that $\pi_1 = 0$). If $\{\pi_i\}$ is any family of disjoint balanced projections, then $\sum \pi_i L_+ \subset L_+$. By finite dimensionality there can be at most a finite number of such π_i . The equality $\sum_1^q \pi_i = I$ for a maximal family follows from the fact that π is balanced iff $I - \pi$ is balanced.

This corollary allows us to assume that in the proof of finite dimensionality of U , the identity is the only nonzero balanced projection. In this case U is "almost" a primary representation as the next lemma shows.

LEMMA 2.4. *U has at most two disjoint central projections.*

PROOF. Let π be a central projection for U . Let H_π be the image of π in H and let $H_{\tilde{\pi}}$ be the closure of the image of H_π under $I(U, U^*)$. Let $\tilde{\pi}$ be the projection in H^* onto $H_{\tilde{\pi}}$. It is clear that $H_{\tilde{\pi}}$ is invariant under U^* so that $\pi \oplus \tilde{\pi} \in \mathcal{Q}'$. We claim also that $\pi \oplus \tilde{\pi} \in \mathcal{Q}$. To see this, it is sufficient to prove that $\pi \oplus \tilde{\pi}$ commutes with $\mathcal{Q}'$. It is obvious that $H_{\tilde{\pi}}$ is invariant under any operator which commutes with U^* , so it suffices to show that $\tilde{\pi}A = A\pi$ for all $A \in I(U, U^*)$ and $\pi A' = A'\tilde{\pi}$ for all $A' \in I(U^*, U)$. The first identity is proven as follows:

$$\tilde{\pi}A = \tilde{\pi}A\pi + \tilde{\pi}A(I - \pi) = A\pi + \tilde{\pi}A(I - \pi).$$

Now, since π is central $U|\pi$ is disjoint from $U|(I - \pi)$ (i.e., they contain no equivalent subrepresentations) and $\tilde{\pi}A(I - \pi) = 0$.

The second identity follows similarly using the inclusion $A'H_\pi \subset H_\pi$, which holds since $A'A$ maps H_π into H_π for all $A \in I(U, U^*)$.

Therefore $\pi \oplus \tilde{\pi}$ and $(\pi)^* \oplus \tilde{\pi}^*$ are central projections. Hence their product $\pi(\tilde{\pi})^* \oplus \pi^*\tilde{\pi}$ is central, so $\pi(\tilde{\pi})^*$ is balanced. Thus $\pi(\tilde{\pi})^* = I$ or 0. If $\pi(\tilde{\pi})^* = I$, then $\pi = I$. If $\pi(\tilde{\pi})^* = 0$, then $\pi + (\tilde{\pi})^*$ is a balanced projection and $\pi + (\tilde{\pi})^* = I$. But if $\sigma \leq \pi$ (i.e., $\sigma\pi = \sigma$), then $(\tilde{\sigma})^* \leq (\tilde{\pi})^*$, so that $\sigma + (\tilde{\sigma})^* = I$. The latter is impossible unless $\sigma = \pi$. Thus if $\pi \neq I$, then π has no smaller central projections. Since π was arbitrary this shows that π and $I - \pi$ are the only central projections for U .

Now, let π be a minimal central projection for U . By the above lemma $I - \pi$ is also such a projection. By restricting U to the image of π we may assume that the only central projection for U is I . Hence U is a primary representation so U must be of the form nU_0 where U_0 is an irreducible representation and $n \in \{1, 2, \dots, \infty\}$. If one shows that U_0 is finite dimensional then it will follow that U is finite dimensional since a cyclic representation cannot contain any finite dimensional representation with infinite multiplicity (see [2, §15.5.3]). Thus we can take $U = U_0$.

We prove that in the irreducible case the set $\mathcal{C}$ (the domain of B_λ) equals H . To see this we use the fact that a topologically irreducible representation of a C^* algebra is algebraically irreducible (cf. [2, §2.8.4]). Let $C^*(G)$ be the group C^* algebra of G . Let U (resp., U^*) denote the representation of $C^*(G)$ corresponding to U (resp., U^*). If $f \in C^*(G)$, then $\tilde{U}(f) \oplus \tilde{U}^*(f) \in \mathcal{Q}$. It follows that $\tilde{U}(f)\varphi \in \mathcal{C}$ for all $f \in C^*(G)$. But the set of $\tilde{U}(f)\varphi$ is invariant under U , and U is irreducible. Hence $\mathcal{C} = H$ as claimed. Since B_λ has an adjoint operator, it follows from the closed graph theorem that B_λ is continuous. As U is irreducible the space $I(U, U^*)$ is at most one dimensional. The mapping $\lambda \rightarrow B_\lambda$ is injective from L' to $I(U, U^*)$, so L' is at most one dimensional. This shows that L_+ has finite codimension in H , which implies the finite dimensionality of H . Thus the proof of the finite dimensionality of U in general is finished.

Since U is finite dimensional, B_λ is a continuous operator from H^* to H^* . The mapping $\lambda \rightarrow B_\lambda$ is injective. It is also surjective, for if $B \in I(U, U^*)$ then $\lambda = B\varphi \in L'$. It is easily seen that $B = B_\lambda$. Thus

$$\dim U = \dim L_+ + \dim L' = \dim L_+ + \dim I_a(U, U^*)$$

and Theorem 2.1 is proven.

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