FUNCTION SPACE FLOW INVARIANCE FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF RETARDED TYPE¹

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ABSTRACT. Let $\mathcal{C}$ denote the Banach space of continuous functions $\phi: [-r, 0] \rightarrow \mathbf{R}^n$, let $\Omega \subset \mathcal{C}$ be closed, and let $f: [0, \infty) \times \Omega \rightarrow \mathbf{R}^n$ be continuous. In this note we establish necessary and sufficient conditions for function space flow invariance for the functional differential equation: $x'(t) = f(t, x_t)$ for $t \geq 0$, $x_0 = \phi \in \Omega$. That is, for each $\phi \in \Omega$ there exist $b > 0$ and a solution $x: [-r, b] \rightarrow \mathbf{R}^n$ such that $x_t \in \Omega$ for each $t \in [0, b]$.

Let $r > 0$ and let $\mathcal{C} = \mathcal{C}([-r, 0]; \mathbf{R}^n)$ denote the Banach space of continuous functions $\phi: [-r, 0] \rightarrow \mathbf{R}^n$ with norm $|\phi| = \max\{|\phi(\theta)|: \theta \in [-r, 0]\}$ and let $\Omega \subset \mathcal{C}$ be closed. If $b > 0$ and $x: [-r, b] \rightarrow \mathbf{R}^n$ is continuous, we define $x_t \in \mathcal{C}$ by $x_t(\theta) = x(t + \theta)$ for each $t \in [0, b]$. Let $f: [0, \infty) \times \Omega \rightarrow \mathbf{R}^n$ be continuous. In this note we establish necessary and sufficient conditions such that for each $\phi \in \Omega$ there exist $b > 0$ and a continuous function $x: [-r, b] \rightarrow \mathbf{R}^n$ satisfying the functional differential equation:

$$x'(t) = f(t, x_t) \quad \text{for } t \in [0, b],$$

$$x_0 = \phi,$$

with $x_t \in \Omega$ for $t \in [0, b]$. Our approach is to construct approximate solutions to a related integral equation employing a modification of the techniques found in Martin [4] and Webb [7]. This function space flow invariance is in the spirit of that defined by Hale [1] as opposed to that investigated by authors such as Leela and Moauro [2] and Seifert [6]. Actually, the invariance considered in [2] and [6] is a special case of that defined above. In remarks following the theorem we indicate the relationship between our results and those of [2] and [6]. Results when $\mathbf{R}^n$ is replaced by a general Banach space are also indicated.

Let $\mathcal{C}$, Ω , and f be as above and $t_0 \in [0, \infty)$. We say that

$$\begin{aligned} x'(t) &= f(t, x_t), & t \geq t_0, \\ x_{t_0} &= \phi \in \Omega, \end{aligned} \tag{*}$$

has a local solution provided there exist $b > 0$ and a continuous function $x: [-r + t_0, t_0 + b] \rightarrow \mathbf{R}^n$ which satisfies (*) with $x_t \in \Omega$ for $t \in [t_0, t_0 + b]$.

We will frequently refer to the functions $F_h: [0, \infty) \times \mathcal{C} \rightarrow \mathcal{C}$ and $T(h)$:

Received by the editors September 4, 1978.

AMS (MOS) subject classifications (1970). Primary 34K05.

¹This work partially supported by NSF Grant SER 77-04218.

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0002-9939/79/0000-0469/\$03.00

$\mathcal{C} \rightarrow \mathcal{C}$ where $h \geq 0$:

$$F_h(t, \phi)\theta = \begin{cases} \phi(h + \theta), & -r \leq \theta \leq -h, \\ \phi(0) + (h + \theta)f(t, \phi), & -h < \theta \leq 0, \end{cases}$$

and

$$[T(h)\phi]\theta = \begin{cases} \phi(h + \theta), & -r \leq \theta \leq -h, \\ \phi(0), & -h < \theta \leq 0. \end{cases}$$

Also, for $\psi \in \mathcal{C}$ we define $|\psi; \Omega| = \inf\{|\psi - \phi|: \phi \in \Omega\}$.

THEOREM. Suppose $\Omega \subset \mathcal{C}$ is closed and $f: [0, \infty) \times \mathcal{C} \rightarrow \mathbf{R}^n$ is continuous. Then the following are equivalent:

- (1) for each $(t_0, \phi) \in [0, \infty) \times \Omega$, (*) has a local solution;
- (2) for each $(t_0, \phi) \in [0, \infty) \times \Omega$, $\liminf_{h \rightarrow 0^+} |F_h(t_0, \phi); \Omega|/h = 0$.

PROOF. The proof rests on the observation that $x: [-r + t_0, t_0 + b] \rightarrow \mathbf{R}^n$ is a solution to (*) if and only if x is continuous and for each $t \in [t_0, t_0 + b]$, x_t satisfies

$$x_t(\theta) = \begin{cases} \phi(t - t_0 + \theta), & -r \leq \theta \leq t_0 - t, \\ \phi(0) + \int_{t_0}^{t+\theta} f(\tau, x_\tau) d\tau, & t_0 - t < \theta \leq 0. \end{cases} \quad (**)$$

That (1) $\rightarrow$ (2) is now clear; for if $(t_0, \phi) \in [0, \infty) \times \Omega$ and (*) has a local solution $x: [-r + t_0, t_0 + b] \rightarrow \mathbf{R}^n$, then

$$\begin{aligned} \lim_{h \rightarrow 0^+} |F_h(t_0, \phi); \Omega|/h &\leq \lim_{h \rightarrow 0^+} |F_h(t_0, \phi) - x_{t_0+h}|/h \\ &= \lim_{h \rightarrow 0^+} \max_{-h < \theta \leq 0} \left| \phi(0) + (h + \theta)f(t_0, \phi) - \phi(0) \right. \\ &\quad \left. - \int_{t_0}^{t_0+h+\theta} f(\tau, x_\tau) d\tau \right|/h \\ &\leq \lim_{h \rightarrow 0^+} \max_{-h < \theta \leq 0} \frac{1}{h} \int_{t_0}^{t_0+h+\theta} |f(t_0, \phi) - f(\tau, x_\tau)| d\tau \\ &= 0. \end{aligned}$$

The proof that (2) $\rightarrow$ (1) given below is obtained via approximate solutions to (**). The construction of the approximate solutions given in the lemma is patterned after that found in Martin [4] and Webb [7].

Let $\phi \in \Omega$, and for simplicity we will suppose $t_0 = 0$. Since f is continuous, there exist positive numbers b , R , and M such that if $(t, \psi) \in [0, b] \times \Omega$ with $|\psi - \phi| \leq R$ then $|f(t, \psi)| \leq M$ and if $|\psi| \leq b(M + 1)$ then $|T(t)\phi - \phi + \psi| \leq R$ for all $t \in [0, b]$.

LEMMA. Let $\Omega \subset \mathcal{C}$ be closed, $f: [0, \infty) \times \Omega \rightarrow \mathbf{R}^n$ be continuous, and $\liminf_{h \rightarrow 0^+} |F_h(t, \psi); \Omega|/h = 0$ for each $(t, \psi) \in [0, \infty) \times \Omega$. Suppose $\phi \in \Omega$ and b , R , and M are as above. Let $\{\epsilon_n\}_1^\infty \subset (0, 1)$ with $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Then for each n there exist $U_n: [0, b] \rightarrow \mathcal{C}$ and a partition $\{t_i^n\}_{i=0}^N$ of $[0, b]$ with

$t_{i+1}^n - t_i^n \leq \varepsilon_n$ such that:

(i) $U_n(0) = \phi$, $U_n(t_i^n) \in \Omega$ with $|U_n(t_i^n) - \phi| \leq R$ and if $t \in [t_i^n, t_{i+1}^n)$ then $U_n(t)$ is defined by

$$[U_n(t)]\theta = \begin{cases} [U_n(t_i^n)](t - t_i^n + \theta), & -r \leq \theta \leq t_i^n - t, \\ [U_n(t_i^n)]0 + (t - t_i^n + \theta)f(t_i^n, U_n(t_i^n)), & t_i^n - t < \theta \leq 0; \end{cases}$$

(ii) $|U_n(t_{i+1}^n) - U_n(t_{i+1}^-)| \leq \varepsilon_n(t_{i+1}^n - t_i^n)$, where $U_n(t_{i+1}^-) = \lim_{t \rightarrow t_{i+1}^-} U_n(t)$;

(iii) if $|\theta - \lambda| \leq t_{i+1}^n - t_i^n$ then $|[U_n(t_i^n)]\theta - [U_n(t_i^n)]\lambda| \leq \varepsilon_n$;

(iv) if $t \in [t_i^n, t_{i+1}^n)$ then $|W_n(t) - U_n(t)| \leq \varepsilon_n t_i^n$, where for $t \in [t_i^n, t_{i+1}^n)$ we define $W_n(t) \in \mathcal{C}$ by

$$[W_n(t)]\theta = \begin{cases} \phi(t + \theta), & -r \leq \theta \leq -t, \\ \phi(0) + \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & -t < \theta \leq 0, \end{cases}$$

and $\gamma_n(\tau) = t_k^n$ when $\tau \in [t_k^n, t_{k+1}^n)$;

(v) if $(t, \psi) \in [t_i^n, t_{i+1}^n) \times \Omega$ with

$$|\psi - U_n(t_i^n)| \leq (t_{i+1}^n - t_i^n)(M + 1) + \max\{|T(h)U_n(t_i^n) - U_n(t_i^n)|: 0 \leq h \leq t_{i+1}^n - t_i^n\}$$

then

$$|f(t, \psi) - f(t_i^n, U_n(t_i^n))| \leq \varepsilon_n \quad \text{and} \quad |T(t - t_i^n)U_n(t_i^n) - U_n(t_i^n)| \leq \varepsilon_n.$$

PROOF. The construction of U_n is by induction; in particular, we suppose U_n is defined on $[0, t_i^n]$. Let δ_i^n be the maximum number satisfying

(a) $\delta_i^n \in [0, \varepsilon_n]$ and if $(t, \psi) \in [t_i^n, t_i^n + \delta_i^n] \times \Omega$ with

$$|\psi - U_n(t_i^n)| \leq \delta_i^n(M + 1) + \max\{|T(h)U_n(t_i^n) - U_n(t_i^n)|: 0 \leq h \leq \delta_i^n\}$$

then

$$|f(t, \psi) - f(t_i^n, U_n(t_i^n))| \leq \varepsilon_n$$

and

$$\max\{|T(h)U_n(t_i^n) - U_n(t_i^n)|: 0 \leq h \leq \delta_i^n\} \leq \varepsilon_n;$$

(b) if $|\theta - \lambda| \leq \delta_i^n$ then $|U_n(t_i^n)\theta - U_n(t_i^n)\lambda| \leq \varepsilon_n$; and

(c) $|F_{\delta_i^n}(t_i^n, U_n(t_i^n)); \Omega| \leq \delta_i^n \varepsilon_n / 2$.

The suppositions on f and the definition of $T(h)$ assure that $\delta_i^n > 0$. Let $t_{i+1}^n = t_i^n + \delta_i^n$ and for $t \in [t_i^n, t_{i+1}^n)$ define $U_n(t) \in \mathcal{C}$ by

$$[U_n(t)]\theta = \begin{cases} [U_n(t_i^n)](t - t_i^n + \theta), & -r \leq \theta \leq t_i^n - t, \\ [U_n(t_i^n)]0 + (t - t_i^n + \theta)f(t_i^n, U_n(t_i^n)), & t_i^n - t < \theta \leq 0. \end{cases}$$

By (c), $|U_n(t_{i+1}^-); \Omega| \leq \delta_i^n \varepsilon_n / 2$ so we may choose $U_n(t_{i+1}^n) \in \Omega$ such that

$$|U_n(t_{i+1}^-) - U_n(t_{i+1}^n)| \leq (t_{i+1}^n - t_i^n) \varepsilon_n.$$

To establish that U_n satisfies (i)–(v) on $[0, t_{i+1}^n]$ we first show that $|W_n(t) -$

$U_n(t) \leq \varepsilon_n t_i^n$ for $t \in [t_i^n, t_{i+1}^n)$. Observe that, for $-r \leq \theta \leq t_i^n - t$,

$$\begin{aligned} T(t - t_i^n) W_n(t_i^n) \theta &= W_n(t_i^n)(t - t_i^n + \theta) \\ &= \begin{cases} \phi(t + \theta), & -r \leq \theta \leq -t, \\ \phi(0) + \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & -t < \theta \leq 0, \end{cases} \end{aligned}$$

and consequently,

$$\begin{aligned} \max_{-r < \theta < t_i^n - t} |W_n(t) \theta - U_n(t) \theta| &= \max_{-r < \theta < t_i^n - t} |T(t - t_i^n) W_n(t_i^n) \theta - T(t - t_i^n) U_n(t_i^n) \theta| \\ &\leq |W_n(t_i^n) - U_n(t_i^n)| \\ &\leq |W_n(t_i^n) - U_n(t_i^{n-})| + |U_n(t_i^{n-}) - U_n(t_i^n)| \\ &\leq \varepsilon_n t_{i-1}^n + \varepsilon_n (t_i^n - t_{i-1}^n) = \varepsilon_n t_i^n. \end{aligned}$$

Also, for $t_i - t < \theta \leq 0$,

$$\begin{aligned} W_n(t_i^n) 0 + (t + \theta - t_i) f(t_i^n, U_n(t_i^n)) &= \phi(0) + \int_0^{t_i^n} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau + \int_{t_i^n}^{t+\theta} f(t_i^n, U_n(t_i^n)) d\tau \\ &= \phi(0) + \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau \\ &= W_n(t) \theta \end{aligned}$$

and thus

$$\begin{aligned} \max_{t_i^n - t < \theta \leq 0} |W_n(t) \theta - U_n(t) \theta| &= |W_n(t_i^n) 0 - U_n(t_i^n) 0| \\ &\leq |W_n(t_i^n) - U_n(t_i^n)| \leq \varepsilon_n t_i^n. \end{aligned}$$

It follows that $|W_n(t) - U_n(t)| \leq \varepsilon_n t_i^n$ for $t \in [t_i^n, t_{i+1}^n)$. We also have

$$\begin{aligned} |U_n(t_{i+1}^n) - \phi| &= |U_n(t_{i+1}^n) - U_n(t_{i+1}^{n-}) + U_n(t_{i+1}^{n-}) \\ &\quad - W_n(t_{i+1}^n) + W_n(t_{i+1}^n) - \phi| \\ &= |\psi + T(t_{i+1}) \phi - \phi| \end{aligned}$$

where

$$\psi(\theta) = \begin{cases} U_n(t_{i+1}^n) \theta - W_n(t_{i+1}^n) \theta, & -r \leq \theta \leq -t_{i+1}^n, \\ U_n(t_{i+1}^n) \theta - W_n(t_{i+1}^n) \theta + \int_0^{t_{i+1}^n + \theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & -t_{i+1}^n < \theta \leq 0, \end{cases}$$

and so

$$\begin{aligned} |\psi| &\leq |U_n(t_{i+1}^n) - U_n(t_{i+1}^{n-})| + |U_n(t_{i+1}^{n-}) - W_n(t_{i+1}^n)| + t_{i+1}^n M \\ &\leq \varepsilon_n (t_{i+1}^n - t_i^n) + \varepsilon_n (t_i^n) + t_{i+1}^n M \\ &\leq t_{i+1}^n (M + 1) \leq b(M + 1). \end{aligned}$$

By the selection of b and M we have

$$|U_n(t_{i+1}^n) - \phi| = |\psi + T(t_{i+1}^n)\phi - \phi| \leq R.$$

It readily follows that U_n satisfies (i)–(v) on $[0, t_{i+1}^n]$. To establish that there exists N such that $t_N^n = b$, suppose for contradiction that $t_i^n < b$ for all i and $\lim_{i \rightarrow \infty} t_i^n = c \leq b$. For each $k = 0, 1, 2, \dots$ define $W_n^k: [t_k^n, c] \rightarrow \mathcal{C}$ by

$$W_n^k(t)\theta = \begin{cases} [U_n(t_k^n)](t - t_k^n + \theta), & -r \leq \theta \leq t_k^n - t, \\ U_n(t_k^n)0 + \int_{t_k^n}^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & t_k^n - t < \theta \leq 0. \end{cases}$$

Using an argument similar to that employed above we have for $t_i^n > t_k^n$ that

$$|W_n^k(t_i^n) - U_n(t_i^n)| \leq \varepsilon_n(t_i^n - t_k^n) \leq \varepsilon_n(c - t_k^n).$$

Also, for $j > l$

$$[W_n^k(t_j) - W_n^k(t_l)]\theta = \begin{cases} [U_n(t_k^n)](t_j^n - t_k^n + \theta) - [U_n(t_k^n)](t_l^n - t_k^n + \theta), & -r \leq \theta \leq t_k^n - t_j^n, \\ U_n(t_k^n)0 + \int_{t_k^n}^{t_j^n+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau \\ \quad - [U_n(t_k^n)](t_l^n - t_k^n + \theta), & t_k^n - t_j^n < \theta \leq t_k^n - t_l^n, \\ \int_{t_k^n}^{t_j^n+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau \\ \quad - \int_{t_k^n}^{t_l^n+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & t_k^n - t_l^n < \theta \leq 0, \end{cases}$$

from which one can show that $\{W_n^k(t_i^n)\}_{i=k}^\infty$ is a Cauchy sequence. It follows that $\lim_{i \rightarrow \infty} U_n(t_i^n)$ exists and since Ω is closed we have $\lim_{i \rightarrow \infty} U_n(t_i^n) = W \in \Omega$. Since $\{U_n(t_i^n)\}_{i=1}^\infty \cup \{W\}$ is compact, by the continuity of f and the definition of $T(h)$ there exists $\delta > 0$ such that if $h \in [0, \delta]$ then h may replace δ_i^n in (a). Also, since $\{U_n(t_i^n)\}_{i=1}^\infty \cup \{W\}$ is compact it is an equicontinuous family so we may assume if $h \in [0, \delta]$ then h may replace δ_i^n in (b). However, there exists $h_0 \in [0, \delta]$ such that $|F_{h_0}(c, W); \Omega| \leq h_0 \varepsilon_n / 3$ and since for sufficiently large i we have $\delta_i^n < h_0$ and δ_i^n is the maximum number satisfying (a)–(c), it must be that $|F_{h_0}(t_i^n, U_n(t_i^n)); \Omega| \geq h_0 \varepsilon_n / 2$. But this is impossible since

$$h_0 \varepsilon_n / 3 \geq |F_{h_0}(c, W); \Omega| = \lim_{i \rightarrow \infty} |F_{h_0}(t_i^n, U_n(t_i^n)); \Omega| \geq h_0 \varepsilon_n / 2.$$

We conclude that there exists N such that $t_N^n = b$. This completes the proof of the lemma.

PROOF OF THE THEOREM (CONTINUED). We now show that (2) $\rightarrow$ (1). We first establish that there exists a continuous function $U: [0, b] \rightarrow \Omega$ such that $\lim_{n \rightarrow \infty} U_n = U$ uniformly on $[0, b]$.

Let W_n be defined by (iv) in the statement of the lemma and write, for $0 < s < t \leq b$,

$$[W_n(t) - W_n(s)]\theta = \begin{cases} \phi(t + \theta) - \phi(s + \theta), & -r \leq \theta \leq -t, \\ \phi(0) + \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau - \phi(s + \theta), & -t < \theta \leq -s, \\ \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau & \\ \quad - \int_0^{s+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau, & -s < \theta \leq 0. \end{cases}$$

With this representation one can show that $\{W_n\}_{n=1}^\infty$ forms an equicontinuous family. To apply Ascoli's theorem to $\{W_n\}_{n=1}^\infty$ we must show for each $t \in [0, b]$ that $\{W_n(t)\}_{n=1}^\infty \subset \mathcal{C}$ is precompact. However, similar to the above one can show for each $t \in [0, b]$ that $\{W_n(t)\}_{n=1}^\infty$ is an equicontinuous family of $\mathcal{C}$ and since $|W_n(t)| \leq |\phi| + bM$ for all $1 \leq n \leq \infty$ we have by Ascoli's theorem that $\{W_n(t)\}_{n=1}^\infty$ is precompact and consequently $\{W_n\}_{n=1}^\infty$ is precompact. Relabelling, if necessary, we have that there exists a continuous function $U: [0, b] \rightarrow \mathcal{C}$ such that $U = \lim_{n \rightarrow \infty} W_n$ uniformly on $[0, b]$. For each $t \in [t_i^n, t_{i+1}^n)$, writing

$$[U_n(t) - U_n(t_i^n)]\theta = \begin{cases} [U_n(t_i^n)](t - t_i^n + \theta) - [U_n(t_i^n)]\theta, & -r \leq \theta \leq t_i^n - t, \\ U_n(t_i^n)0 + (t - t_i^n + \theta)f(t_i^n, U_n(t_i^n)) - U_n(t_i^n)\theta, & t_i^n - t < \theta \leq 0, \end{cases}$$

and using (iii) of the lemma, we obtain $|U_n(t) - U_n(t_i^n)| \leq \varepsilon_n(1 + M)$. Also, for $t \in [t_i^n, t_{i+1}^n)$ by (iv) we see $|W_n(t) - U_n(t)| \leq \varepsilon_n t_i^n \leq \varepsilon_n b$ and thus $U(t) = \lim_{n \rightarrow \infty} U_n(t) = \lim_{n \rightarrow \infty} U_n(\gamma_n(t)) \in \Omega$ uniformly on $[0, b]$. The continuity of f and compactness of $\{(t, U(t)): 0 \leq t \leq b\}$ yields

$$\lim_{n \rightarrow \infty} f(\gamma_n(t), U_n(\gamma_n(t))) = f(t, U(t))$$

uniformly on $[0, b]$. Let $t \in [0, b]$. Then, if $\theta \in [-r, t]$,

$$U(t)\theta = \lim_{n \rightarrow \infty} W_n(t)\theta = \phi(t + \theta)$$

and, if $\theta \in [-t, 0]$,

$$\begin{aligned} U(t)\theta &= \lim_{n \rightarrow \infty} W_n(t)\theta = \lim_{n \rightarrow \infty} \phi(0) + \int_0^{t+\theta} f(\gamma_n(\tau), U_n(\gamma_n(\tau))) d\tau \\ &= \phi(0) + \int_0^{t+\theta} f(\tau, U(\tau)) d\tau. \end{aligned}$$

Defining

$$x(t) = \begin{cases} U(0)t, & -r \leq t \leq 0, \\ U(t)0, & 0 < t \leq b, \end{cases}$$

we have that x is a solution to $(*)$ with $x_t = U(t) \in \Omega$ for each $t \in [0, b]$. This completes the proof of the theorem.

REMARK 1. Let $\Lambda \subset \mathbf{R}^n$ and $\Omega(\Lambda) \subset \mathcal{C}$ be defined by $\Omega(\Lambda) = \{\phi \in \mathcal{C}: \phi(\theta) \in \Lambda \text{ for } \theta \in [-r, 0]\}$. Note that for $t \geq 0$, $x_t \in \Omega(\Lambda)$ if and only if $x(s) \in \Lambda$ for $s \geq -r$. Also, in the case that Λ is convex (2) of the theorem is equivalent to

(2)' for each $(t, \phi) \in [0, \infty) \times \Omega(\Lambda)$, $\liminf_{h \rightarrow 0^+} |\phi(0) + hf(t, \phi); \Lambda|/h = 0$.

That (2) $\rightarrow$ (2)' is evident. Suppose (2)' holds, let $\varepsilon > 0$, and let $(t, \phi) \in [0, \infty) \times \Omega(\Lambda)$. By (2)' there exist $h > 0$ and $x_h \in \Lambda$ such that $|\phi(0) + hf(t, \phi) - x_h| \leq h\varepsilon/2$. Define $\psi_h \in \Omega(\Lambda)$ by

$$\psi_h(\theta) = \begin{cases} \phi(\theta + h), & -r \leq \theta \leq -h, \\ [(\theta + h)x_h - \theta\phi(0)]/h, & -h < \theta \leq 0. \end{cases}$$

Then

$$\begin{aligned} \frac{1}{h} |F_h(t, \phi); \Omega(\Lambda)| &\leq \frac{1}{h} |F_h(t, \phi) - \psi_h| \\ &= \max_{-h < \theta \leq 0} \frac{1}{h} \left| [\phi(0) + (h + \theta)f(t, \phi)] - \left[\frac{(\theta + h)x_h - \theta\phi(0)}{h} \right] \right| \\ &\leq \max_{-h < \theta \leq 0} \frac{1}{h} \left(1 + \frac{|\theta|}{h} \right) |\phi(0) + hf(t, \phi) - x_h| \leq \varepsilon \end{aligned}$$

and it follows that (2)' $\rightarrow$ (2). Consequently, the techniques in the theorem yield results in Seifert [6]. Actually, in [6], $\mathcal{C}$ is replaced by $\mathcal{CB} = \{\phi: [-\infty, 0] \rightarrow \mathbf{R}^n: \phi \text{ is continuous and bounded}\}$. However, the necessary modifications to the theorem are easily performed.

REMARK 2. Suppose we replace $\mathbf{R}^n$ with a general Banach space $\bar{X}$. This generalization requires no change in the lemma. The difficulty arises in the proof of the theorem in which we show (2) $\rightarrow$ (1) by obtaining the convergence of a subsequence of $\{W_n\}_{n=1}^\infty$; in particular, we are not assured for each $(t, \theta) \in [0, b] \times [-t, 0]$ that $\{W_n(t)\theta\}_{n=1}^\infty$ is precompact. This can obviously be overcome if we assume that f maps bounded subsets of $[0, b] \times \Omega$ into precompact subsets of $\bar{X}$. We may also obtain that $\{W_n(t)\theta\}_{n=1}^\infty$ is precompact by supposing that

(3) f is uniformly continuous on $[0, b] \times \Omega$ and for each bounded set $P \subset \Omega$ and

$$\begin{aligned} \lim_{h \rightarrow 0^-} (\alpha[\{\psi(0) + hf(t, \psi)\}_{\psi \in P}] - \alpha[\{\psi(0)\}_{\psi \in P}]) / h \\ \leq g(t, \alpha[\{\psi(0)\}_{\psi \in P}]) \end{aligned}$$

whenever $\alpha[\{\psi(\theta)\}_{\psi \in P}] \leq \alpha[\{\psi(0)\}_{\psi \in P}]$ for $\theta \in [-r, 0]$. Here $g: [0, \infty) \times [0, \infty) \rightarrow \mathbf{R}$ is such that one obtains an estimate on solutions to $x'_-(t) \leq g(t, x(t))$; for a bounded set $A \subset \bar{X}$, $\alpha[A]$ denotes the measure of noncompactness of A ; and $\alpha[\{\psi(0)\}_{\psi \in P}] = \alpha[\{\psi(0): \psi \in P\}]$. Assumption (3) is made by Leela and Moauro [2] and is motivated by a similar condition used for ordinary differential equations by Li [3] and Martin [5]. The resulting

proof that $\{W_n(t)\theta\}_{n=1}^\infty$ is precompact is straightforward given the techniques used in those papers. In [2], $\Omega = \Omega(\Lambda)$ for some closed $\Lambda \subset \bar{X}$ and the invariance assumption made on f is

$$(4) \liminf_{h \rightarrow 0^+} |\phi(0) + hf(t, \phi); \Lambda|/h = 0$$

for all $(t, \phi) \in [0, \infty) \times \mathcal{C}$ with $\phi(0) \in \Lambda$. The following simple example typifies a situation in which our results apply but those of [2] do not and so our results actually improve those of [2].

Let $\Lambda = [-1, 0]$ and $f: \mathcal{C} \rightarrow \mathbf{R}$ be defined by $f(\phi) = \phi(-r)\phi(0)$. It is easy to verify that for $\phi \in \Omega(\Lambda)$, $F_h(\phi) \in \Omega(\Lambda)$ for sufficiently small h and so (2) holds. However, if $\phi(\theta) = -(2/r)\theta - 1$ then $\phi(0) = -1 \in \Lambda$ (note that $\phi \notin \Omega(\Lambda)$) but $f(\phi) = -1$ and thus

$$\liminf_{h \rightarrow 0^+} \frac{|\phi(0) + hf(\phi); \Lambda|}{h} = \liminf_{h \rightarrow 0^+} \frac{|-1 - h; \Lambda|}{h} = 1.$$

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