PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 1, October 1979

UNIFORM CLOSURES OF FOURIER-STIELTJES ALGEBRAS

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ABSTRACT. Let H be a closed normal subgroup of a locally compact group G. Assume that f is a continuous function on G such that it is constant on the cosets of H in G and it can be approximated uniformly by coefficient functions of unitary representations of G. We show that f can be approximated uniformly by coefficient functions of representations of G which are lifted from unitary representations of G/H. For abelian G, our theorem is a conjecture of R. B. Burckel.

Let G be a locally compact group and $B(G)$ its Fourier-Stieltjes algebra. Then $B(G)$ is a commutative Banach algebra with its usual norm $\|\cdot\|$ (cf. Eymard [**4**, p. 197]). Let $C(G)$ be the algebra of bounded complex-valued continuous functions on G with sup norm $\|\cdot\|_\infty$. The uniform closure of a set E in $C(G)$ will be denoted by E^-.

For a closed normal subgroup H of G, denote the canonical homomorphism of G onto G/H by π. Let $\tilde{\pi}$: $C(G/H) \to C(G)$ be defined by $\tilde{\pi}f = f \circ \pi, f \in C(G/H)$. The purpose of this paper is to prove the following.

THEOREM. *Let G be a locally compact group and H a closed normal subgroup of G. Then*

$$\tilde{\pi}(B(G/H)^-) = \tilde{\pi}(C(G/H)) \cap B(G)^-.$$

Recall that $B(G)$ is the algebra of coefficient functions of continuous unitary representations of G or, equivalently, the algebra of linear combinations of positive definite continuous functions on G (see [**4**]). Therefore what we stated in the Abstract is equivalent to the above theorem. If G is abelian with dual group Γ and if $M(\Gamma)$ is the Banach algebra of bounded regular Borel measures on Γ then $B(G) = \{\hat{\mu}: \mu \in M(\Gamma)\}$ where $\hat{\mu}(x) = \int_\Gamma(\gamma, x)\, d\mu(\gamma)$, $x \in G$ and $\|\hat{\mu}\| = \|\mu\|$. In this case the above theorem can be stated as follows: Suppose f is a continuous function on G such that it is constant on the cosets $H + x$ and it can be approximated uniformly by Fourier-Stieltjes transforms of measures on Γ then it can be approximated uniformly by Fourier-Stieltjes transforms of measures on the annihilator of H in Γ. This is a conjecture given by Burckel in his monograph [**1**, p. 81, Problem 7]. He was able to prove it if H is further assumed to be compact.

Received by the editors October 12, 1978.

AMS (MOS) subject classifications (1970). Primary 43A30; Secondary 22D10, 43A07.

Key words and phrases. Locally compact group, Fourier-Stieltjes algebra, uniform limit, invariant mean, weakly almost periodic function.

[1]Supported in part by NSF grant MSC 76-06377A01

PROOF OF THE THEOREM. Eymard proved in [4, p. 202] that

$$\tilde{\pi}(B(G/H)) = \tilde{\pi}(C(G/H)) \cap B(G). \tag{1}$$

Since π is onto, $\tilde{\pi}$ is an isometry with respect to the sup norm. Therefore, by (1), $\tilde{\pi}(B(G/H)^{-}) \subset \tilde{\pi}(C(G/H)) \cap B(G)^{-}$.

To see the converse, let $f \in \tilde{\pi}(C(G/H)) \cap B(G)^{-}$. Then, for a given $\varepsilon > 0$, there exists $h \in B(G)$ such that

$$|f(x) - h(x)| < \varepsilon, \qquad x \in G. \tag{2}$$

By Ryll-Nardzewski's fixed point theorem, $W(H)$, the algebra of continuous weakly almost periodic functions on H with sup norm, has a unique translation invariant mean m: $m \in W(H)^*$, $\|m\| = 1$, $m \geqslant 0$ and $m(k \cdot t) = m(k)$ if $k \in W(H)$ and $t \in H$ where $k \cdot t \in W(H)$ is defined by $(k \cdot t)(t') = k(t't)$, $t' \in H$ (cf. [1, p. 15]). For $x \in G$, let $h_x \in C(H)$ be defined by $h_x(t) = h(tx)$, $t \in H$. It is easy to check that $h_x \in B(H)$. Since $B(H) \subset W(H)$ (cf. [1, p. 36]) $h_1(x) = m(h_x)$ is defined for each $x \in G$. We claim that

$$h_1 \in \tilde{\pi}(B(G/H)). \tag{3}$$

By (1), it suffices to show that (i) $h_1 \in \tilde{\pi}(C(G/H))$ and (ii) $h_1 \in B(G)$.

(i) Since functions in $B(G)$ are uniformly continuous, if $x_\alpha \to x$ in G then

$$|h_1(x_\alpha) - h_1(x)| = |m(h_{x_\alpha} - h_x)| \leqslant \|h_{x_\alpha} - h_x\|_\infty \to 0.$$

So h_1 is continuous on G. Since m is H-invariant, $h_1(tx) = m(h_x \cdot t) = m(h_x) = h_1(x)$, for $t \in H$ and $x \in G$. Therefore, $h_1 \in \tilde{\pi}(C(G/H))$.

(ii) We will apply the following result of Davis [3, Theorem 5.1]: There exists a net of open and relatively compact subsets U_α of H such that

$$m(k) = \lim_\alpha \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} k(t)\, d\lambda(t), \qquad k \in W(H). \tag{4}$$

Here λ is a fixed left Haar measure of H. (For locally compact amenable groups, this fact is well known; see [5, p. 43].) For each α, set $\varphi_\alpha = (1/\lambda(U_\alpha)) \cdot \chi_{U_\alpha}$ where χ_{U_α} is the characteristic function of U_α in H. Let $d\mu_\alpha = (1/\Delta)\varphi_\alpha^{\sim}\, d\lambda$ where Δ is the modular function on H and $\varphi_\alpha^{\sim}(t) = \varphi_\alpha(t^{-1})$, $t \in H$. Then $\|\mu_\alpha\| = \int (1/\Delta)\varphi_\alpha^{\sim}\, d\lambda = 1$. We will consider μ_α as a measure on G. Set

$$\xi_\alpha(x) = \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} h(tx)\, d\lambda(t), \qquad x \in G.$$

Then $\xi_\alpha = \mu_\alpha * h$. By [4, p. 198], $\xi_\alpha \in B(G)$ and $\|\xi_\alpha\| \leqslant \|\mu_\alpha\|\, \|h\| = \|h\|$. By (4), $h_1(x) = m(h_x) = \lim_\alpha \xi_\alpha(x)$, $x \in G$. Therefore, h_1 is a pointwise limit of a net of functions ξ_α in $B(G)$ with $\|\xi_\alpha\| \leqslant \|h\|$. Since h_1 is continuous, by [4, p. 202], $h_1 \in B(G)$. The proof of (ii) is completed.

Note that for each $t \in H$, $f_x(t) = f(xt) = f(x)$. Therefore, $m(f_x) = f(x)$. Now, by (2), $\|f_x - h_x\|_\infty < \varepsilon$ and hence $|m(f_x) - m(h_x)| < \varepsilon$, $x \in G$. So, we have

$$|f(x) - h_1(x)| < \varepsilon, \qquad x \in G. \tag{5}$$

By assumption there exists $g \in C(G/H)$ such that $\tilde{\pi} g = f$ and by (3) there exists $h_2 \in B(G/H)$ such that $\tilde{\pi}(h_2) = h_1$. So (5) can be written as

$$|g(Hx) - h_2(Hx)| < \varepsilon, \qquad Hx \in G/H.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $g \in B(G/H)^-$ and hence $f = \tilde{\pi}(g) \in \tilde{\pi}(B(G/H)^-)$.

Remark. If, in the above theorem, H is compact, then the fact that $h_1 \in B(G)$, the crux of our proof, becomes obvious. For, in this case, $h_1 = \lambda * h$ where λ is the normalized Haar measure of H. If G is abelian, H is compact, i.e., the case considered by Burckel [**1**, Theorem A.39], and $h = \hat{\mu} \in B(G)$ then $h_1 = \hat{\mu}_1$ where μ_1 is defined by $\mu_1(B) = \mu(B \cap \Lambda)$, B a Borel set in Γ and Λ the annihilator of H in Γ:

$$\begin{aligned} h_1(x) &= \int_H h(x + t)\, d\lambda(t) \\ &= \int_\Gamma (\gamma, x)\left(\int_H (\gamma, t)\, d\lambda(t)\right) d\mu(\gamma) \\ &= \int_\Lambda (\gamma, x)\, d\mu(\gamma) = \hat{\mu}_1(x), \qquad x \in G. \end{aligned}$$

Note that Burckel proved his Theorem A.39 by applying Ramirez' characterization theorem for $B(G)^-$ (cf. [**1**, Theorem A.38]). He then applied his Theorem A.39 to give a proof of the following theorem of Ramirez [**6**]: If G is a noncompact locally compact abelian group then $B(G)^- \subsetneq W(G)$ (cf. [**1**, p. 67]).

Let G and H be as in our theorem. It is known that $\tilde{\pi}(W(G/H)) = \tilde{\pi}(C(G/H)) \cap W(G)$ (see [1]). Therefore we have the following.

Corollary 1. *If $B(G)^- = W(G)$ then $B(G/H)^- = W(G/H)$.*

By combining this corollary with Ramirez' theorem in [**6**] we can state the following.

Corollary 2. *Suppose G is a locally compact group with a noncompact locally compact abelian quotient group. Then $W(G) \supsetneq B(G)^-$.*

For example, $W(G) \supsetneq B(G)^-$ if G is the Heisenberg group or the $ax + b$ group. Note that there exist noncompact groups G with $W(G) = B(G)^-$, see [**2**].

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