A MAXIMAL INEQUALITY FOR H^1 -FUNCTIONS ON A GENERALIZED WALSH-PALEY GROUP

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ABSTRACT. Let $G = \prod_{i=1}^{\infty} Z(p_i)$ be the countable product of discrete cyclic groups of order p_i . We assume that $\sup_{i \geq 1} p_i < \infty$. We consider Walsh-Fourier series on G and define H^1 -functions on G by the Coifman-Weiss atoms. Let $K_n(x)$ be the n th $(C, 1)$ -kernel. We prove that $\int_G \sup_{n \geq 1} |(K_n * f)(x)| d\mu < C \|f\|_{H^1}$. Here $d\mu$ is the normalized Haar measure, $\|\cdot\|_{H^1}$ is the H^1 -norm and C is a constant independent of f .

1. Introduction. Let $G = \prod_{i=1}^{\infty} Z(p_i)$ be the countable product of discrete cyclic groups $Z(p_i)$ of order p_i , $p_i \geq 2$, and $p \equiv \sup_{i \geq 1} p_i < \infty$. Set $m_0 = 1$ and $m_s = \prod_{i=1}^{s-1} p_i$, $s = 1, 2, \dots$. Let $\varphi_s(x) = \exp(2\pi i x^{(s)}/p_s)$, $s = 0, 1, 2, \dots$, where $x = \{x^{(s)}\}_{s=0}^{\infty} \in G$ and $x^{(s)} = 0, 1, \dots, p_s - 1$. We express each nonnegative integer n as a finite sum $n = \sum_{s=0}^{\infty} \alpha_s m_s$, $0 \leq \alpha_s < p_s$, and define $\chi_n = \prod_{s=0}^{\infty} \varphi_s^{\alpha_s}$. The functions $\{\chi_n\}_{n=0}^{\infty}$ are the characters of G and form a complete orthonormal system on G (see N. Ja. Vilenkin [7], J. Gosselin [4] and M. H. Taibleson [6]).

Let $G_0 = G$ and $G_s = \prod_{i=0}^{s-1} \{0\} \times \prod_{j=s}^{\infty} Z(p_j)$, $s = 1, 2, \dots$. Then G_s is the subgroup of G and $\mu(G_s) = m_s^{-1}$, where μ is the normalized Haar measure. ω_s denotes any coset of G_s , that is, $\omega_s = x + G_s$ for some $x \in G$.

Let $D_n = \sum_{j=0}^{n-1} \chi_j$, $n = 1, 2, \dots$, be the n th Dirichlet kernel. It is well known that

$$D_{m_s}(x) = \begin{cases} m_s & \text{if } x \in G_s, \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(see Vilenkin [7]).

We define the atomic H^1 on G in the manner of R. R. Coifman and G. Weiss [1]. A function $a(x)$ is called an atom when $a(x)$ is the constant function 1 or $a(x)$ satisfies the conditions:

- (i) the support of a is contained in an "interval" $\omega_N = x_0 + G_N$,
- (ii) $|a(x)| \leq \mu(\omega_N)^{-1}$ for all $x \in G$, and
- (iii) $\int_G a(x) d\mu = 0$.

A function f is said to belong to H^1 if $f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)$, $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, where the a_j are atoms. If f belongs to H^1 , then f also belongs to $L^1(G)$. We define the H^1 -norm by $\|f\|_{H^1} = \inf \{ \sum_{j=1}^{\infty} |\lambda_j|; f = \sum_{j=1}^{\infty} \lambda_j a_j(x), \text{ the } a_j \text{ are atoms} \}$.

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Let $f^*(x) = \sup_{s \geq 0} |(D_{m_s} * f)(x)|$. Then Coifman and Weiss [1, pp. 611–612] showed that an L^1 -function f belongs to H^1 if and only if f^* is integrable; moreover $\|f\|_{H^1}$ is equivalent to $\|f^*\|_{L^1}$.

Let $K_n(x) = (1/n) \sum_{j=1}^n D_j(x)$, $n = 1, 2, \dots$, be the n th $(C, 1)$ -kernel. For the Walsh-Fourier series, S. Yano [8] and F. Schipp [5], obtained the weak type inequality; that is, there is a constant C such that $\text{meas}\{x \in [0, 1]: \sup_{n \geq 1} |(K_n * f)(x)| > \alpha\} \leq (C/\alpha) \|f\|_{L^1}$ for all f and $\alpha > 0$. In this note we shall prove the following “strong type” inequality for H^1 -functions on our group G .

THEOREM. *There is a constant C , independent of f , such that for all $f \in H^1$,*

$$\int_G \sup_{n \geq 1} |(K_n * f)(x)| d\mu \leq C \|f\|_{H^1}. \quad (2)$$

2. Some lemmas.

LEMMA 1. *Let $n = \sum_{s=0}^M \alpha_s m_s$. Then*

$$|K_n| \leq \frac{p+1}{n} \sum_{s=0}^M \alpha_s m_s D_{m_s} + \frac{1}{n} \sum_{s=0}^M \alpha_s m_s |K_{m_s}|. \quad (3)$$

PROOF. We write $n = \alpha_M m_M + l$, $0 \leq l < m_M$. Then it follows from the definition of K_n that

$$nK_n = \alpha_M m_M K_{\alpha_M m_M} + l D_{\alpha_M m_M} + l \varphi_M^{\alpha_M} K_l. \quad (4)$$

We have also that

$$D_{\alpha_M m_M} = D_{m_M} \sum_{j=0}^{\alpha_M-1} \varphi_M^j, \quad (5)$$

and

$$\alpha_M m_M K_{\alpha_M m_M} = m_M \sum_{j=1}^{\alpha_M-1} D_{j m_M} + m_M K_{m_M} \sum_{j=1}^{\alpha_M-1} \varphi_M^j. \quad (6)$$

Using (4), (5) and (6) inductively, we obtain that¹

$$\begin{aligned} nK_n = \sum_{j=0}^M \chi_{\sum_{t=j+1}^M \alpha_t m_t} \left[\left\{ m_j \sum_{r=0}^{\alpha_j-2} (\alpha_j - 1 - r) \varphi_j^r \right. \right. \\ \left. \left. + \sum_{t=0}^{j-1} \alpha_t m_t \sum_{r=0}^{\alpha_t-1} \varphi_j^r \right\} D_{m_j} + m_j \sum_{r=0}^{\alpha_j-1} \varphi_j^r K_{m_j} \right]. \end{aligned} \quad (7)$$

Then (3) follows from (7) immediately.

¹ The sums $\sum_{s=M+1}^M \alpha_s m_s$ and $\sum_{s=0}^{-1} \alpha_s m_s$ are taken to be 0.

LEMMA 2. Let

$$L_{m_s}(x) = \sum_{j=1}^s \frac{m_j}{p_{j-1}^2 m_s} \sum_{l=0}^{p_{j-1}-2} (p_{j-1} - 1 - l) \varphi_{j-1}^l(x) \sum_{k=1}^{p_{j-1}-1} D_{m_s}(x + kg_{j-1}),$$

where

$g_j = (0, \dots, 0, 1, 0, \dots)$ (the 1 is in the j th place) and

$$kg_j = g_j + \dots + g_j \text{ (} k \text{ times)}.$$

Then

$$K_{m_s}(x) = (1/2 + 1/2m_s) \cdot D_{m_s}(x) + L_{m_s}(x). \quad (8)$$

PROOF. By exchanging the order of summation, we have

$$m_s K_{m_s} = m_{s-1} \left\{ \sum_{l=0}^{p_{s-1}-1} (p_{s-1} - 1 - l) \varphi_{s-1}^l \right\} D_{m_{s-1}} + m_{s-1} \sum_{l=0}^{p_{s-1}-1} \varphi_{s-1}^l K_{m_{s-1}}. \quad (9)$$

From (1), we have also

$$D_{m_s}(x) = \frac{1}{p_s} \sum_{k=0}^{p_s-1} D_{m_{s+1}}(x + kg_s). \quad (10)$$

By induction, using (9) and (10), we obtain (8).

For Lemmas 1 and 2, refer to N. J. Fine [2], [3] and S. Yano [8].

LEMMA 3. Let $a(x)$ be a function supported by an interval ω_N . We define a maximal function a^{**} of a by $a^{**}(x) = \sup_{s \geq N} (D_{m_s} * |a|)(x)$. If $|a(x)| \leq \mu(\omega_N)^{-1}$ for $x \in \omega_N$, then

$$\int_G a^{**}(x) d\mu \leq 1. \quad (11)$$

PROOF. If $x \notin \omega_N$ and $y \in \omega_N$, then $x - y \notin G_N$, so $x - y \notin G_s$ for $s \geq N$. Therefore, if $x \notin \omega_N$ and $s \geq N$,

$$(D_{m_s} * |a|)(x) = \int D_{m_s}(x - y) |a(y)| d\mu(y) = 0.$$

Thus (11) is immediate.

3. Proof of Theorem. From the definition of H^1 , in order to prove (2) it suffices to show that there is a constant C such that

$$\int_G \sup_{n \geq 1} |(K_n * a)(x)| d\mu \leq C \quad (12)$$

for all atoms a .²

By Lemmas 1 and 2, we obtain that

$$\sup_{n \geq 1} |(K_n * a)(x)| \leq C_0 \|a\|_\infty \quad \text{for all } x \in G. \quad (13)$$

² Throughout this note the constants $C, C_0, C_1, \dots$ depend only on p .

Therefore we have the inequality (12) with $C = C_0$ if $a \equiv 1$. Thus we may assume that an atom a satisfies (i), (ii) and (iii). Since a is supported by ω_N and satisfies (iii), it follows that

$$\hat{a}(l) \equiv \int_G a(x) \bar{\chi}_l(x) d\mu = 0 \quad \text{for all } l < m_N,$$

and moreover,

$$(D_l * a)(x) = \sum_{j=0}^{l-1} \hat{a}(j) \chi_j(x) = 0 \quad \text{for all } l \leq m_N \text{ and } x \in G.$$

Thus, if $1 \leq n \leq m_N$, then $K_n * a \equiv 0$. By this reason we assume from now that n is a positive integer greater than m_N .

Now we divide G into the $N + 1$ domains ω_N and $\omega_{N-(r+1)} \setminus \omega_{N-r}$, $r = 0, 1, 2, \dots, N-1$, where $\omega_s = x_0 + G_s$, $s = 0, 1, \dots, N$. Let us estimate $K_n * a$ pointwise on each domain. If $x \in \omega_N$ then we obtain from (ii) and (13) that

$$|(K_n * a)(x)| \leq C_0 \mu(\omega_N)^{-1}. \quad (14)$$

Let $x \in \omega_{N-(r+1)} \setminus \omega_{N-r}$ and $n = \sum_{s=0}^M \alpha_s m_s$. By Lemmas 1 and 2, we have

$$\begin{aligned} |(K_n * a)(x)| &\leq \frac{p+2}{n} \sum_{s=0}^M \alpha_s m_s (D_{m_s} * |a|)(x) \\ &\quad + \frac{1}{n} \sum_{s=0}^M \alpha_s m_s (|L_{m_s}| * |a|)(x). \end{aligned} \quad (15)$$

Since $x \notin \omega_{N-r}$, we have $x - y \notin G_{N-r}$ for $y \in \omega_N$. Hence, by (1), $(D_{m_s} * |a|)(x) = 0$ for all $s \geq N - r$. On the other hand, from (i), (ii) and (1), we have, for $s < N - r$,

$$(D_{m_s} * |a|)(x) \leq m_s \int |a| d\mu \leq m_s.$$

Thus, for the first summand of the right-hand side of (15) we have that

$$\begin{aligned} \frac{p+2}{n} \sum_{s=0}^M \alpha_s m_s (D_{m_s} * |a|)(x) &\leq \frac{p+2}{n} \sum_{s=0}^{N-(r+1)} \alpha_s m_s m_s \\ &\leq (p+2) m_{N-(r+1)} \sum_{s=0}^{N-(r+1)} \alpha_s \\ &\leq (p+2) m_{N-(r+1)} \frac{m_{N-r}}{m_N}. \end{aligned} \quad (16)$$

Next we shall estimate the second summand of the right-hand side of (15). If

$0 \leq s < N - r$, then

$$\begin{aligned} (|L_{m_s}| * |a|)(x) &\leq \sum_{j=1}^s \frac{m_j}{m_s} \sum_{k=1}^{p_{j-1}-1} \int D_{m_s}(x - y + kg_{j-1}) |a(y)| d\mu(y) \\ &\leq pm_s \sum_{j=1}^s \frac{m_j}{m_s} \leq C_1 m_{N-r}. \end{aligned} \quad (17)$$

If $s \geq N - r$ and $j \neq N - r$ ($1 \leq j \leq s$), then $x - y + kg_{j-1} \notin G_s$ for all $y \in \omega_N$ because $x - y \in G_{N-(r+1)} \setminus G_{N-r}$ and $kg_{j-1} \in G_{j-1} \setminus G_j$. Therefore, for $s \geq N - r$,

$$(|L_{m_s}| * |a|)(x) \leq \frac{m_{N-r}}{m_s} \sum_{k=1}^{p_{N-r-1}-1} \int D_{m_s}(x - y + kg_{N-r-1}) |a(y)| d\mu(y). \quad (18)$$

Moreover, if $s \geq N$, we have

$$(|L_{m_s}| * |a|)(x) \leq \frac{m_{N-r}}{m_s} \sum_{k=1}^{p_{N-r-1}} a^{**}(x + kg_{N-r-1}). \quad (19)$$

By (17), (18) and (19),

$$\begin{aligned} \frac{1}{n} \sum_{s=0}^M \alpha_s m_s (|L_{m_s}| * |a|)(x) &\leq C_2 \frac{m_{N-r}}{n} \sum_{s=0}^{N-(r+1)} \alpha_s m_s \\ &+ \frac{m_{N-1}}{n} \sum_{s=N-r}^{N-1} \frac{\alpha_s m_s}{m_s} \sum_{k=1}^{p_{N-r-1}} \int D_{m_s}(x - y + kg_{N-r-1}) |a(y)| d\mu \\ &+ \frac{\sum_{s=N}^M \alpha_s m_s m_{N-r}}{nm_N} \sum_{k=1}^{p_{N-r-1}} a^{**}(x + kg_{N-r-1}) \\ &\leq C_2 m_{N-r} \frac{m_{N-r}}{m_N} + p \frac{m_{N-r}}{m_N} \sum_{s=N-r}^{N-1} \sum_{k=1}^{p_{N-r-1}} \\ &\quad \cdot \int D_{m_s}(x - y + kg_{N-r-1}) |a(y)| d\mu(y) \\ &+ \frac{m_{N-r}}{m_N} \sum_{k=1}^{p_{N-r-1}} a^{**}(x + kg_{N-r-1}). \end{aligned} \quad (20)$$

From (16) and (20) we obtain that, for all $x \in \omega_{N-(r+1)} \setminus \omega_{N-r}$,

$$\begin{aligned} \sup_{n \geq 1} |(K_n * a)(x)| &\leq C_3 \frac{m_{N-r}}{m_N} \left\{ m_{N-(r+1)} + \sum_{s=N-r}^{N-1} \right. \\ &\quad \cdot \sum_{k=1}^{p_{N-r-1}} (D_{m_s} * |a|)(x + kg_{N-r-1}) \\ &\quad \left. + \sum_{k=1}^{p_{N-r-1}} a^{**}(x + kg_{N-r-1}) \right\}. \end{aligned} \quad (21)$$

By (14) and (21),

$$\begin{aligned} \int_G \sup_{n \geq 1} |(K_n * a)(x)| d\mu &\leq C_0 \int_{\omega_N} \mu(\omega_N)^{-1} d\mu \\ &+ C_3 \sum_{r=0}^{N-1} \frac{m_{N-r}}{m_N} \int_{\omega_{N-r-1} \setminus \omega_{N-r}} \left[m_{N-r-1} + \sum_{s=N-r}^{N-1} \sum_{k=1}^{p_{N-r-1}} \right. \\ &\quad \cdot (D_{m_s} * |a|)(x + kg_{N-r-1}) \\ &\quad \left. + \sum_{k=1}^{p_{N-r-1}} a^{**}(x + kg_{N-r-1}) \right] d\mu(x). \end{aligned}$$

From Lemma 3 and the fact that $\int (D_{m_s} * |a|)(x) d\mu \leq 1$, we obtain that

$$\int_G \sup_{n \geq 1} |(K_n * a)(x)| d\mu \leq C_0 + C_3 \sum_{r=0}^{N-1} \frac{m_{N-r}}{m_N} (1 + pr + p) \leq C_4.$$

The last constant C_4 is independent of a and N . This completes the proof.

If $f \in L^1(G)$, we know that $|f(x)| \leq \sup_{n \geq 1} |(K_n * f)(x)|$ and $|f(x)| \leq f^*(x)$ for almost everywhere x in G (see [3], [8] and [4, p. 350]). So it is interesting whether $\int_G \sup_{n \geq 1} |(K_n * f)(x)| d\mu$ is equivalent to $\|f\|_{H^1}$. But we have not yet obtained any result for the converse direction of our theorem.

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