CONSTRUCTION OF A RIGID ARONSZAJN TREE¹

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ABSTRACT. We construct in ZFC an Aronszajn tree with no automorphism.

Using the nonisomorphic Aronszajn trees constructed by Gaifman and Specker [1] we build in ZFC a normal rigid Aronszajn tree thus answering a question of T. J. Jech [2].

For a tree T and $a \in T$, Ta denotes the part of T above a , i.e., $Ta = \{x \in T \mid x \geq a\}$. T_α , $\alpha < \omega_1$, is the α th level of T . We say that $a, b \in T$ meet at the level α if there is an $e \in T_\alpha$, $e < a$, $e < b$, but no such e exists in $T_{\alpha+1}$.

By [1] we have for every uncountable $X \subseteq \omega_1$ a normal Aronszajn tree $T(X)$ such that:

- (1) for every $a \in T(X)$ there is an uncountable subset of $T(X)_a$ every two elements of which meet at level in X , and
- (2) there does not exist an uncountable subset of $T(X)$ every two elements of which meet at level in $\omega_1 - X$.

(Briefly, if we start from an Aronszajn tree $T \subseteq \omega^\omega$ then $T(X) \subseteq \omega^\omega$ is the set of all functions f such that the collapse of $f|X$ is in T and $f|(\omega_1 - X)$ is different from zero only finitely many times.)

We will construct a normal Aronszajn tree R such that there is no order preserving function $f: R \rightarrow R$ other than the identity. Let $X, X_{\alpha,n}$, $\alpha < \omega_1$, $n < \omega$, be pairwise disjoint uncountable subsets of ω_1 , construct the trees $T(X_{\alpha,n})$ such that their domain will be pairwise disjoint. We construct a sequence R^n , $n < \omega$, of Aronszajn trees such that $R^n \subseteq R^{n+1}$ by induction on n . A point in $R^{n+1} - R^n$ is said to be of rank $n + 1$.

R^0 is $T(X)$. Suppose R^n is defined, let $\{a_\gamma^n \mid \gamma < \omega_1\}$ be its elements, for every a_γ^n , $a_\gamma^n \in R_\alpha^n$ for some $\alpha < \omega_1$, take any member b_γ^n of $T(X_{\gamma,n})_{\alpha+1}$ and plant $T(X_{\gamma,n})_{b_\gamma^n}$ above a_γ^n . The result is R^{n+1} . Again, R^{n+1} is obtained by partial ordering $R^n \cup \bigcup_{\gamma < \omega_1} T(X_{\gamma,n})_{b_\gamma^n}$ as follows: for elements of R^n or $T(X_{\gamma,n})$ the order is as in R^n or $T(X_{\gamma,n})$, respectively, b_γ^n is above a_γ^n and the partial order is transitive. For $x \in T(X_{\gamma,n})$ we define $r(x) = b_\gamma^n$ and $r^*(x) = a_\gamma^n$, for $x \in R^0$, $r(x)$ is the root of R^0 .

$R = \bigcup_{n < \omega} R^n$ is a normal Aronszajn tree. Note that for $m < n$ an element of rank n cannot be below an element of rank m . We prove that R is

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rigid. Let $f: R \rightarrow R$ be order preserving and for some distinct $x, y \in R_\alpha$, $f(x) = y$.

For some $n < \omega$, $x, y \in R^n$, $x = a_\gamma^n$ for some $\gamma < \omega_1$ and $T(X_{\gamma,n})_{b_\gamma^n}$ was grafted above a_γ^n , it follows from (1) that we can find $\aleph_1$ elements in $R_{f(x)}$ every two of which meet at level in $X_{\gamma,n}$. Let $k < \omega$ be the least integer such that we can find an uncountable $A \subseteq R_y^k$ every two elements of which meet at level in $X_{\gamma,n}$. Look at $B = \{r^*(x) | x \in A\}$ for such an A . If it is uncountable then we have $\aleph_1$ elements in R_y^{k-1} contradicting the definition of k . Hence we can find b such that $b = r(x)$ for uncountably many x in A . b is compatible with y , hence distinct from x so that $b = b_\eta^l$ for some (η, l) distinct from (γ, n) it follows that we have found $\aleph_1$ elements in $T(X_{\eta,l})$ every two of which meet at level outside $X_{\eta,l}$, and this is a contradiction.

REFERENCES

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