THE MINIMAL HARMONIC FUNCTIONS OF SOJOURN PROCESSES OF CERTAIN FINITE STATE MARKOV CHAINS

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ABSTRACT. To a finite state, discrete time Markov chain a new chain is constructed which counts the number of sojourns in each state. This sojourn process is a transient Markov chain. The minimal harmonic functions of this chain are identified.

1. Introduction. Let I be a finite set of s states and $Z_n, n \in \mathbb{N}$ be a sequence of independent identically distributed I -valued random variables and let $p_i = \Pr(Z_n = i)$. To Z_n we associate a sequence of random variables $X_n = (\chi_i^{(n)}, \dots, \chi_s^{(n)})$ with values in $\mathbb{N}_0^s$ ($\mathbb{N}_0$ the set of nonnegative integers) by

$$\chi_i^{(n)} = \sum_{m=1}^n 1_i(Z_m).$$

Clearly, X_n is a transient Markov chain. D. Blackwell and D. Kendall calculated in [1] the Martin boundary of this chain. (See also [5, p. 394].) (Actually, Blackwell and Kendall calculated the Martin boundary of Pólya's urn scheme, but it turns out that Pólya's urn scheme is a h -path-transformation of X_n .) The boundary can be identified with the simplex

$$\Delta_s = \left\{ t = (t_1, \dots, t_s): 0 \leq t_i, \sum_{i=1}^s t_i = 1 \right\}.$$

The minimal nonnegative harmonic function corresponding to $t \in \Delta_s$ is given by

$$\mathbb{N}_0^s \ni (x_1, \dots, x_s) \rightarrow \prod_{i=1}^s (t_i/p_i)^{x_i}.$$

It is the purpose of this paper to generalize this result to the case where Z_n is a finite state Markov chain with all transition probabilities nonvanishing. To such a chain we associate X_n as above, but one has to take the joint variables $Y_n = (X_n, Z_n)$ to get a Markov chain. We call Y_n the sojourn process of Z_n . The minimal nonnegative harmonic functions again correspond bijective to points in Δ_s , but the connection is more complicated.

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2. Notations and statements. Let $I = \{1, \dots, s\}$ be as above and $P = (p_{ij})_{i,j \in I}$ be a matrix of transition probabilities on I which all are nonvanishing. Let $\delta_i \in \mathbb{N}_0^s$ have components 0 except the j th which is 1. To P we associate a transition probability matrix Q on $\tilde{I} = \mathbb{N}_0^s \times I$: if $x, y \in \mathbb{N}_0^s$ then

$$q_{(x,i),(y,j)} = \begin{cases} p_{ij} & \text{if } y = x + \delta_j, \\ 0 & \text{else.} \end{cases}$$

We call a function $h: I \rightarrow \mathbb{R}^+$ harmonic if for all $(x, i) \in I$,

$$h(x, i) = \sum_{j \in I} p_{ij} h(x + \delta_j, j) = \sum_{(y,i) \in \tilde{I}} q_{(x,i),(y,j)} h(y, j).$$

(We shall only consider nonnegative harmonic functions.) Let $\mathcal{H}$ denote the set of harmonic functions which are not identically 0. A harmonic function h is called minimal if whenever $h' \in \mathcal{H}$ and $h' \leq h$ (i.e., $h'(x, i) \leq h(x, i)$ for all $(x, i) \in \tilde{I}$) then there is a constant $c > 0$ with $h' = ch$. Let $\mathcal{H}_e$ be the set of minimal harmonic functions.

We shall give a complete description of the elements in $\mathcal{H}_e$.

If $u = (u_1, \dots, u_s) \in (\mathbb{R}^+)^s - \{0\}$ let $v = (v_1, \dots, v_s)$ be defined by $v_i = \sum_j p_{ij} u_j$. u and v will always be connected in this way. Clearly, from the assumption that $p_{ij} > 0$ for all $i, j \in I$, we have $v_i > 0$ for all i . To $u \in (\mathbb{R}^+)^s - \{0\}$ we associate a function $h_u: \tilde{I} \rightarrow \mathbb{R}^+$ by $h_u(x, i) = v_i \prod_{j=1}^s (u_j / v_j)^{x_j}$.

THEOREM 1. $\mathcal{H}_e = \{h_u: u \in (\mathbb{R}^+)^s - \{0\}\}$.

The proof is deferred to §3.

We consider now the subsets $\mathcal{D} = \{h \in \mathcal{H}: h(0, 1) = 1\}$ and $\mathcal{D}_e = \mathcal{D} \cap \mathcal{H}_e$. It is easy to see that for $h \in \mathcal{H}$, $h(0, 1) > 0$. (In fact, if $h(0, 1) = 0$ then from $h(0, 1) = \sum_j p_{1j} h(\delta_j, j)$ one obtains $h(\delta_j, j) = 0$ for all j and then $h(0, j) = 0$ for all j and a straightforward conclusion gives $h = 0$.) So each $h \in \mathcal{H}$ has a unique representation $h = ch'$ where $c > 0$ and $h' \in \mathcal{D}$.

If $\mathcal{H}$ is equipped with the topology of pointwise convergence, $\mathcal{D}$ is a compact convex subset of $\mathbb{R}^{\tilde{I}}$ with $\mathcal{D}_e$ as the set of extremal points. Now $h_u \in \mathcal{D}$ if and only if $\sum_{j=1}^s p_{1j} u_j = 1$. Let $\tilde{\Delta}_s = \{u \in (\mathbb{R}^+)^s: \sum_{j=1}^s p_{1j} u_j = 1\}$, then the topology on $\mathcal{D}_e$ corresponds to the usual topology on $(\mathbb{R}^+)^s$ under the correspondence $\tilde{\Delta}_s \ni u \mapsto h_u \in \mathcal{D}_e$. From general Choquet representation theory one thus obtains (see e.g. [2]):

THEOREM 2. To any $h \in \mathcal{D}$ there exists a unique probability measure μ on the Borel sets of $\tilde{\Delta}_s$ such that

$$h(x, i) = \int h_u(x, i) \mu(du)$$

for all $(x, i) \in I$.

One would like to have a probabilistic interpretation of the elements in $\tilde{\Delta}_s$. Let us first fix some notations: Let $\Omega = I^{\mathbb{N}_0}$ equipped with the product σ -algebra which we denote by $\mathfrak{A}$. For $i \in I$ let P^i be the law on $(\Omega, \mathfrak{A})$ which

governs the Markov chain with transition probabilities p_{ij} and starting point $i \in I$. Let Z_n , $n \in \mathbb{N}_0$, be the projections of Ω on its factors and let X_n be defined as in the introduction ($X_0 = 0$). Then $Y_n = (X_n, Z_n)$ is a transient Markov chain with state space $\tilde{I}$ and transition probabilities q_{ij} . We call Y_n , $n \in \mathbb{N}_0$ the sojourn process. Actually we can start Y_n in any point $(x, i) \in \tilde{I}$ simply by replacing X_n by $x + X_n$ and starting Z_n in i . We denote by $Q^{(x, i)}$ the thus induced law on $\tilde{I}^{\mathbb{N}_0}$. To $u \in \tilde{\Delta}_s$ we associate Markov chain transition probabilities $p_{ij}^u = p_{ij}u_j/v_i$. Let $\pi^u = (\pi_1^u, \dots, \pi_s^u)$ be the stationary distribution of the p_{ij}^u , that is π^u is the unique element in Δ_s (Δ_s defined as in the introduction) with $\pi_j^u = \sum_{i=1}^s \pi_i^u p_{ij}^u$. One thus obtains a mapping $\Psi: \tilde{\Delta}_s \rightarrow \Delta_s$ defined by $\Psi(u) = \pi^u$.

PROPOSITION 3. Ψ is a homeomorphism.

The proof is deferred to §3.

It is now easy to see that the h -path transformation (see, e.g., [5]) of the Y_n process by h_u is exactly the sojourn process associated with the transition probabilities p_{ij}^u . So to any probability measure μ on Δ_s there corresponds exactly one element in $\mathcal{D}_e$, namely $h_{\Psi^{-1}(\mu)}$, such that the h -path transformation of the Y_n process with this element is the sojourn process of a chain with stationary distribution μ .

The present work has relations to the investigations of harmonic functions and Martin boundaries of the so called space-time process (Z_n, n) , where Z_n , $n \in \mathbb{N}_0$ are recurrent Markov chains (much more general than ours, see [6]). Indeed, n is simply $\sum_{i=1}^s \chi_i^{(n)}$ in our notation. So our sojourn processes obviously carry more information than the space-time processes.

There is also a connection with certain conditionings investigated by Darroch and Seneta [3]. Let I_0 be a proper nonvoid subset of I and let $T = \inf\{n \in \mathbb{N}: Z_n \notin I_0\}$. Then it is easy to see that $Z_0, \dots, Z_n$ conditioned on $\{T > n\}$ is a nonhomogeneous Markov chain. Darroch and Seneta have shown that for $n \rightarrow \infty$ the transition probabilities of this chain converge to stationary transition probabilities q_{ij} on I_0 , that is, for $i \in I_0$ and each $k \in \mathbb{N}_0$,

$$\lim_{n \rightarrow \infty} P^i(X_{k+1} = j | X_k = i, T > n) = q_{ij}.$$

One can show that $q_{ij} = p_{ij}^u$ where $u = \Psi^{-1}(\mu)$ where μ is a probability on I_0 such that the Donsker-Varadhan information $I(\mu)$ (see [4]) is minimal among probability measures which are carried by I_0 .

Now $T > n$ is simply a condition on the sojourn times of Z_n : $\sum_{i \notin I_0} \chi_i^{(n)} = 0$. One may consider other conditionings: Let Δ_0 be a closed subset of Δ_s and $l \in I$ such that $Q^{(0, l)}(X_n/n \in \Delta_0) > 0$ for sufficiently large n . Then one can take elementary conditionings on $\{X_n/n \in \Delta_0\}$. In general, $Z_0, \dots, Z_n$ is not a Markov chain, but $Y_0, \dots, Y_n$ is. One may conjecture that, as in the Darroch-Seneta case, if $I(\mu)$ attains a minimum on Δ_0 which is uniquely

attained in μ_0 , then

$$\lim_{n \rightarrow \infty} Q^{(0,l)}(Y_{k+1} = (y,j) | Y_k = (x,i), X_n/n \in \Delta_0) = q_{(x,i),(y,j)}^u$$

where $q_{(x,i),(y,j)}^u$ are constructed as above from the p_{ij}^u and where $u = \Psi^{-1}(\mu_0)$. As far as I know, such conditionings have not been considered in the literature as yet.

3. Proofs. For $x \in \mathbb{N}_0^2$ we associate a map $\phi_x: \mathcal{H} \rightarrow \mathcal{H}$ by $(\phi_x h)(y, i) = h(x + y, i)$. It is obvious that $\phi_x h \in \mathcal{H}$ if $h \in \mathcal{H}$. We simply write ϕ_i for ϕ_{δ_i} . For $i \in I$ let $\tau_i: (\tilde{I})^{\mathbb{N}_0} \rightarrow \bar{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$ be defined by

$$\tau_i((x_0, i_0), (x_1, i_1), \dots) = \inf\{n \in \mathbb{N}_0: x_{n,i} \geq 1\}$$

where $x_{n,i}$ is the i th component of x_n . Clearly τ_i is a stopping time for the Y_n process and $Q^{(x,j)}(\tau_i < \infty) = 1$ for all $(x, j) \in \tilde{I}$ and $Q^{(x,j)}(\tau_i = 0) = 1$ if $x_i \geq 1$. For $h \in \mathcal{H}$ and $i \in I$ let $\Lambda_i h$ be the function $\tilde{I} \rightarrow \mathbb{R}^+$ defined by

$$(\Lambda_i h)(x, j) = E^{(x,j)}(h(Y_{\tau_i} - \delta_i))$$

where $(y, l) - \delta_i = (y - \delta_i, l)$.

LEMMA 1. *If $h \in \mathcal{H}$ then $\Lambda_i h \in \mathcal{H}$.*

PROOF. Let $(x, l) \in \tilde{I}$,

$$\begin{aligned} \sum_{(y,j) \in \tilde{I}} q_{(x,l)(y,j)} (\Lambda_i h)(y, j) \\ = \sum_{j \in I} p_{lj} E^{(x+\delta_j, l)} h(Y_{\tau_i} - \delta_i) = E^{(x,l)} h(Y_{\hat{\tau}_i} - \delta_i) \end{aligned}$$

where $\hat{\tau}_i = \max(\tau_i, 1)$. Now $x_i = 0$ if and only if $\tau_i \geq 1$ $Q^{(x,l)}$ a.s. So if $x_i = 0$ the above expression becomes

$$E^{(x,l)} h(Y_{\tau_i} - \delta_i) = (\Lambda_i h)(x, l)$$

and if $x_i \geq 1$ it equals $E^{(x,l)} h(Y_1 - \delta_i)$ which by harmonicity is

$$h((x - \delta_i, l)) = E^{(x,l)} h(Y_0 - \delta_i) = E^{(x,l)} (Y_{\tau_i} - \delta_i) = (\Lambda_i h)(x, l).$$

So the lemma is proved.

Clearly, one has for $h \in \mathcal{H}$,

$$(\Lambda_i \phi_i h)(x, j) = E^{(x,j)} h(Y_{\tau_i}) \leq h(x, j) \quad (3.1)$$

the last inequality because $h(Y_n)$, $n \in \mathbb{N}_0$ is a nonnegative martingale and τ_i is a stopping time.

$$\phi_i \Lambda_i h = h. \quad (3.2)$$

LEMMA 2. *If $h \in \mathcal{H}_e$ then $\phi_i h = 0$ or $\phi_i h \in \mathcal{H}_e$.*

PROOF. From (3.1), $\Lambda_i \phi_i h \leq h$ so $\Lambda_i \phi_i h = ch$ for some $c \in [0, 1]$. If $c < 1$ then it follows from the fact that $(\Lambda_i \phi_i h)(x, j) = h(x, j)$ for all (x, j) with $x_i \geq 1$ that $h(x, j) = 0$ for $x_i \geq 1$, so $\Lambda_i \phi_i h = 0$ and therefore from (3.2), $\phi_i h = 0$. If $c = 1$, let $h' \in \mathcal{H}$, $h' \leq \phi_i h$. It follows that $h = \Lambda_i \phi_i h \geq \Lambda_i h'$ so

there is a $c' \in [0, 1]$ with $\Lambda_i h' = ch$ and so $h' = \phi_i \Lambda_i h' = c' \phi_i h$, and it is proved that $\phi_i h$ is extremal.

LEMMA 3. Let $h \in \mathcal{H}_e$ and $x, y \in \mathbb{N}_0^s$ with $\sum_{i \in I} x_i y_i = 0$ and $\phi_x h \neq 0$ and $\phi_y h \neq 0$. Then $\phi_{x+y} h \neq 0$.

PROOF. We proceed by induction on $|x| = \sum_{i \in I} x_i$. $|x| = 0$ is clear. If i is such that $y_i = 0$ and $\sum x_i y_i = 0$ and $\phi_{x+y} h \neq 0$ then

$$\Lambda_i \phi_{x+\delta_i+y} h = \phi_y \phi_x \Lambda_i \phi_i h = \phi_{y+x} h$$

by the proof of Lemma 2. So $\phi_{x+\delta_i+y} h \neq 0$.

LEMMA 4. $1 \in \mathcal{H}_e$.

PROOF. Let $h \in \mathcal{H}$, $h \leq 1$. Fix a state $i \in I$ and let $\tilde{h}_i: \mathbb{N}_0^s \rightarrow \mathbb{R}^+$ be defined by $\tilde{h}_i(x) = h(x, i)$. Let $\tau_0 = 0$ and τ_k be the time of the k th hitting of i by the chain Z_n , $n \in \mathbb{N}$. It is well known that $X_{\tau_k} - X_{\tau_{k-1}}$, $k \in \mathbb{N}$ are independent identically distributed $\mathbb{N}_0^s$ valued random variables under the law $Q^{(y, i)}$ for every $y \in \mathbb{N}_0^s$. As $\{h(Y_n), n \in \mathbb{N}_0\}$ is a bounded martingale it follows that $\{\tilde{h}_i(X_{\tau_k}), k \in \mathbb{N}_0\}$ is a bounded martingale, so $\tilde{h}_i$ is bounded harmonic for the random walk X_{τ_k} , $k \in \mathbb{N}_0$. Our assumptions on the p_{ij} clearly imply that the support of the law of $X_{\tau_1} - X_0$ generates Z^s as an Abelian group. So it follows from the Choquet-Deny theorem (see [7, p. 135]) that $\tilde{h}_i$ is a constant, say c_i . From harmonicity of h it follows that $c_i = \sum_{j \in I} p_{ij} c_j$ for all $i \in I$, so all c_i are the same. So h is constant and it follows that $1 \in \mathcal{H}_e$.

PROOF OF THEOREM 1. The proof splits in two parts:

(I) For each $u \in (\mathbb{R}^+)^s - \{0\}$, $h_u \in \mathcal{H}_e$.

(II) If $h \in \mathcal{H}_e$ there is an $u \in (\mathbb{R}^+)^s - \{0\}$ with $h = h_u$.

Proof of (I). Let $u \in (\mathbb{R}^+)^s - \{0\}$ and $N_u = \{i \in I: u_i = 0\}$. For $i, j \in N_u$ let $\tilde{p}_{ij} = p_{ij} u_j / v_i \neq 0$. The $\tilde{p}_{ij}$; $i, j \in N_u^c$ form a Markov transition probability matrix on N_u^c .

Let $h \in \mathcal{H}$, $h \leq h_u$. Clearly $h(x, i) = 0$ if $x_j \geq 1$ for some $j \in N_u$. It is then easy to see that h/h_u restricted to elements (x, i) , $i \notin N_u$ and $x_j = 0$ for $j \in N_u$ is bounded harmonic for the space time process associated to the $\tilde{p}_{ij}$. From Lemma 4 (applied to $\tilde{p}$) there is a $c \in (0, 1]$ with $h(x, i) = c h_u(x, i)$ for all (x, i) satisfying the above restriction but clearly then for all $(x, i) \in \tilde{I}$ as for the other elements both sides are 0. So h_u is minimal.

Proof of (II). Let $h \in \mathcal{H}_e$. It is easy to see that $h(0, l) \neq 0$ for each $l \in I$. Indeed if $h(0, l) = 0$ for some l h would be identically 0.

Let $N_h = \{i \in I: \phi_i h = 0\}$ and $d = \sum_{i \in N_h^c} \delta_i$. From Lemma 3, $\phi_d h \neq 0$. Let $i \notin N_h$ and $i_1, \dots, i_k$ be an enumeration of the elements of N_h^c with $i_k = i$. Then for $x \in \mathbb{N}_0^s$

$$h(x, i) \geq p_{ii_1} p_{i_1 i_2} \dots p_{i_{k-1} i_k} h(x + d, i).$$

It follows that there is a $c > 0$ such that $h(x, i) \geq c h(x + d, i)$ for all x and $i \notin N_h$. But then for all $(x, j) \in \tilde{I}$,

$$h(x, j) = \sum_{i \in N_h^c} p_{ji} h(x + \delta_i, i) > c \sum_{i \in N_h^c} p_{ji} h(x + d + \delta_i, i) = ch(x + d, i)$$

that is $h > c\phi_d h$. As $\phi_d h \neq 0$ there is a $c_1 > 0$ such that $h = c_1 \phi_d h$ and then $h = c_1^2 \phi_d^2 h$.

Let now $l, i \notin N_h^c$ and let $i_1, \dots, i_{2k-1}$ be a sequence of elements in N_h^c where each element of N_h^c appears exactly twice except i which appears once and further $i_{2k-1} = l$. Then

$$(\phi_i h)(x, l) = h(x + \delta_i, l) \geq p_{li} p_{i i_1} \cdots p_{i_{2k-2} i_{2k-1}} h(x + 2d, l)$$

for each $x \in N_0^s$. So it follows as above that there is a $c_2 > 0$ with $\phi_i h \geq c_2 \phi_d^2 h$. From $h = c_1^2 \phi_d^2 h$ it follows that $\phi_d^2 h \neq 0$. So from Lemma 2 it follows that there is a constant $\lambda_i > 0$ with $\phi_i h = \lambda_i h$. This obviously remains true for $i \in N_h$ and $\lambda_i = 0$.

It follows that for $(x, l) \in \tilde{I}$, $h(x, l) = \lambda_1^{x_1} \lambda_2^{x_2} \cdots \lambda_s^{x_s} h(0, l)$. We further have the relation

$$h(0, i) = \sum_{j \in I} p_{ij} h(\delta_j, j) = \sum_{j \in I} p_{ij} \lambda_j h(0, j)$$

so putting $u_i = \lambda_i h(0, i)$, $u = (u_1, \dots, u_s)$ one easily obtains $h = h_u$.

PROOF OF PROPOSITION 3. As $\tilde{\Delta}_s$ and Δ_s both are compact, it suffices to show that Ψ is continuous and bijective. Clearly the p_{ij}^u depend continuously on $u \in \tilde{\Delta}_s$ and the expectation of the recurrence times depend continuously on the transition probabilities. So the coefficients π_i^u of the stationary distribution being the inverses of the expected recurrence times depend continuously on u . We now prove that Ψ is onto. First, if $\pi \in \Delta_s$ has all components different from 0 it follows from Lemma 2.4 of [4] that there is a $u \in \tilde{\Delta}_s$ with $\pi = \pi^u$. If now π has some components 0, say $\pi_1 = \cdots = \pi_r = 0$ ($1 \leq r < s$). Then let $(\tilde{p}_{ij})_{i,j=r+1, \dots, s}$ be defined by

$$\tilde{p}_{ij} = \frac{p_{ij}}{\sum_{k=r+1}^s p_{ik}}$$

and let $u = (u_{r+1}, \dots, u_s)$ with $\sum_{j=r+1}^s p_{1j} u_j = 1$ be such that p_{ij}^u has stationary distribution $(\pi_{r+1}, \dots, \pi_s)$ on $\{r+1, \dots, s\}$. Then $u = (0, \dots, 0, u_{r+1}, \dots, u_s)$ does it.

It remains to show that Ψ is 1-1. Let $u, u' \in \Delta_s$ with $\pi^u = \pi^{u'}$. We may assume that $u_i \neq 0$ for all $i \in I$. Now $p_{ij}^{u'} = (p_{ij}^u)^{(u'/u)}$ where $(u'/u)_i = u'_i/u_i$. So it follows from Lemmas 2.5 and 2.7 of [4] that

$$\sum_j \log \left((u'_j/u_j) \left(\sum_k p_{jk}^u (u'_k/u_k) \right)^{-1} \right) \pi_j^u = 0$$

or

$$\sum_j \pi_j^u \log(u'_j/u_j) = \sum_j \pi_j^u \log \left(\sum_k p_{jk}^u (u'_k/u_k) \right).$$

Now if $u \neq u'$ then $\log(\sum_k p_{jk}^u(u'_k/u_k)) > \sum_k p_{jk}^u \log(u'_k/u_k)$ so

$$\begin{aligned} \sum_j \pi_j^u \log(u'_j/u_j) &> \sum_j \pi_j^u \sum_k p_{jk}^u \log(u'_k/u_k) \\ &= \sum_k \pi_k^u \log(u'_k/u_k) \end{aligned}$$

which is a contradiction.

REFERENCES

1. D. Blackwell and D. G. Kendall, *The Martin boundary of Pólya's scheme and an application to stochastic population growth*, J. Appl. Probability **1** (1964), 248–296.
2. G. Choquet, *Lectures on analysis*, Benjamin, Reading, Massachusetts, 1969.
3. J. N. Darroch and E. Seneta, *On quasi-stationary distributions in absorbing discrete-time finite Markov chains*, J. Appl. Probability **2** (1965), 88–100.
4. M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time*, Comm. Pure Appl. Math. **28** (1975), 1–47.
5. J. G. Kemeny, J. L. Snell and A. W. Knapp, *Denumerable Markov chains*, Springer, New York, 1976.
6. J. Lamperti and J. L. Snell, *Martin boundaries for certain Markov chains*, J. Math. Soc. Japan **15** (1963), 113–128.
7. D. Revuz, *Markov chains*, North-Holland, Amsterdam, 1975.

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