EXPLICIT QUANTIZATION OF THE KEPLER MANIFOLD

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ABSTRACT. Any representation π of $\mathrm{SO}(2, 4)$ quantizing the Kepler manifold has the same lowest highest weight as the representation ν_0 in the Sternberg-Wolf description of the $U(2, 2)$ -restriction of the metaplectic representation of $\mathrm{Sp}(4; \mathbf{R})$. Hence, modulo covering groups, π is unitarily equivalent to ν_0 .

0. Introduction. The Kepler manifold $T^+(S^3)$ is the cotangent bundle of the 3-sphere, minus the zero section, with the symplectic structure induced by that of the cotangent bundle. It is a Hamiltonian symplectic homogeneous space of the conformal group $\mathrm{SO}(2, 4)$ (cf. [11]). The action of $\mathrm{SO}(2, 4)$ on $T^+(S^3)$ has been quantized by various authors ([3], [6], [7], [10], [11]) to give an irreducible unitary representation π of $\mathrm{SO}(2, 4)$ on $L^2(S^3)$. However, these constructions of π suffer from being either ad hoc or else arrived at by a limiting procedure. Here, we give an explicit identification of π within the framework of the metaplectic representation.

In this note we show that π is essentially unitarily equivalent to a certain representation ν_0 of the 2-sheeted cover $\mathrm{SU}(2, 2)$ of $\mathrm{SO}(2, 4)$, described in [9]. It was noted in [12] that the coadjoint orbit of $\mathrm{SU}(2, 2)$ which corresponds to ν_0 under the moment map [4] is the coadjoint orbit of $\mathrm{SO}(2, 4)$ symplectomorphic to $T^+(S^3)$, which of course suggests, but does not prove, that $\pi = \nu_0$. Here we prove this equivalence by examining restrictions to a maximal compact subgroup of $\mathrm{SU}(2, 2)$. The interpretation of this result in terms of geometric quantization will be the subject of another paper.

1. The representations π and π_1 . Rawnsley [8] and Blattner [1] have discussed two positive polarizations of the symplectic manifold $T^+(S^3)$ and their corresponding Hilbert spaces. The first polarization F is just the cotangent fibration, and the associated Hilbert space $\mathcal{H}_F$ is naturally isomorphic to $L^2(S^3)$. The second polarization, G , is obtained as follows: Identify $T^+(S^3)$ with

$$\{(e, x) \in \mathbf{R}^4 \times \mathbf{R}^4: e \cdot e = 1, e \cdot x = 0, x \neq 0\}. \quad (1)$$

Send (e, x) to $|x|e + ix \in \mathbf{C}^4$. The image is

$$X = \{z \in \mathbf{C}^4: z \cdot z = 0, z \neq 0\}, \quad (2)$$

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X is then a Kähler manifold with Kähler form

$$-2^{-5/2} id(|z|^{-1} \{z \cdot d\bar{z} - \bar{z} \cdot dz\}).$$

This structure defines G . The associated Hilbert space $\mathcal{H}_G$ consists of holomorphic functions on X square integrable with respect to the measure

$$\exp(-4\pi|x|)2^{5/2}|x|^{1/2}\gamma, \quad (3)$$

where γ is the Liouville measure on X .

There is no positive polarization of $T^+(S^3)$ stable under $\mathrm{SO}(2, 4)$ [11]. However F is $\mathrm{SO}(1, 4)$ stable and G is $\mathrm{SO}(2) \times \mathrm{SO}(4)$ stable, and hence geometric quantization [2] provides unitary representations π_F of $\mathrm{SO}(1, 4)$ on $\mathcal{H}_F$ and π_G of $\mathrm{SO}(2) \times \mathrm{SO}(4)$ on $\mathcal{H}_G$. Moreover, the half form pairing of $\mathcal{H}_F$ with $\mathcal{H}_G$ (see [8]) gives a bounded nonunitary operator T of $\mathcal{H}_F$ onto $\mathcal{H}_G$, with bounded inverse, which intertwines $\pi_F|_{\mathrm{SO}(4)}$ and $\pi_G|_{\mathrm{SO}(4)}$.

For our purposes π will be any irreducible unitary representation of $\mathrm{SO}(2, 4)$ such that $\pi|_{\mathrm{SO}(1, 4)} \cong \pi_F$ and $\pi|_{\mathrm{SO}(2) \times \mathrm{SO}(4)} \cong \pi_G$.

As usual, the indefinite unitary group $\mathrm{U}(2, 2) = \{g \in \mathbf{C}^{4 \times 4} : ghg^* = h\}$, where

$$h = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

and $\mathrm{SU}(2, 2) = \{g \in \mathrm{U}(2, 2) : \det g = 1\}$. Now $\bigwedge^2(\mathbf{C}^4) = \mathbf{C}^6$ has a real form $\mathbf{R}^6$ invariant under $\{\bigwedge^2(g) : g \in \mathrm{SU}(2, 2)\}$. This action of $\mathrm{SU}(2, 2)$ on $\mathbf{R}^6$ preserves a nondegenerate quadratic form of signature $(2, 4)$. In this way we get a homomorphism $\alpha : \mathrm{SU}(2, 2) \rightarrow \mathrm{SO}(2, 4)$, where $\alpha(g) = \bigwedge^2(g)|_{\mathbf{R}^6}$, and this α is in fact a double covering. Letting $\pi_1 = \pi \circ \alpha$, we obtain an irreducible unitary representation of $\mathrm{SU}(2, 2)$.

2. The representation ν_0 . Fix a nondegenerate antisymmetric bilinear form $\{u, v\}$ on $\mathbf{R}^8$. The symplectic group $\mathrm{Sp}(4; \mathbf{R})$ is the automorphism group of $(\mathbf{R}^8, \{\cdot, \cdot\})$. If $u, v \in \mathbf{R}^8$, then $\xi_{u,v} : x \mapsto \frac{1}{2}(\{u, x\}v + \{v, x\}u)$ belongs to the Lie algebra $\mathfrak{sp}(4; \mathbf{R})$. Fix a basis $p_1, \dots, p_4, q_1, \dots, q_4$ of $\mathbf{R}^8$ with $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\{p_j, q_k\} = \delta_{jk}$. Then $\mathfrak{sp}(4; \mathbf{R})$ has basis

$$\xi_{a,b} = \xi_{p_a, p_b}, \quad \xi'_{a,b} = \xi_{p_a, q_b}, \quad \xi''_{a,b} = \xi_{q_a, q_b}. \quad (4)$$

Let λ be Lebesgue measure on $\mathbf{C}^4$. We have a Hilbert space

$$\mathcal{H} = \left\{ f : \mathbf{C}^4 \rightarrow \mathbf{C} \text{ holomorphic} : \int |f(z)|^2 \exp(-|z|^2) d\lambda < \infty \right\} \quad (5)$$

with inner product

$$\langle f_1, f_2 \rangle = \pi^{-4} \int f_1(z) \overline{f_2(z)} \exp(-|z|^2) d\lambda.$$

In multi-index notation $z^n = z_1^{n_1} \cdots z_4^{n_4}$, $n! = n_1! \cdots n_4!$, where $n = (n_1, \dots, n_4)$, the $\varphi_n(z) = z^n / \sqrt{n!}$ form an orthonormal basis of $\mathcal{H}$.

The metaplectic group $\mathrm{Mp}(4; \mathbf{R})$ is the two-sheeted covering group of

$\mathrm{Sp}(4; \mathbf{R})$. It has a unitary representation μ on $\mathcal{H}$, called the *metaplectic representation*, specified by

$$\begin{aligned} d\mu(\xi_{a,b}) &= -\frac{i}{2}(\partial_a \partial_b - z_a \partial_a - z_b \partial_a + z_a z_b - \delta_{a,b}), \\ d\mu(\xi'_{a,b}) &= \frac{1}{2}(\partial_a \partial_b - z_a \partial_b + z_b \partial_a - z_a z_b), \\ d\mu(\xi''_{a,b}) &= \frac{i}{2}(\partial_a \partial_b + z_a \partial_b + z_b \partial_a + \delta_{a,b}), \end{aligned} \quad (6)$$

where $\partial_a = \partial/\partial z_a$ (see [9]).

Now $\mathrm{U}(2, 2)$ is naturally isomorphic to the subgroup of $\mathrm{Sp}(4; \mathbf{R})$ with Lie algebra spanned by the

$$\begin{aligned} \xi_{a,b} + \xi''_{a,b}, & \quad 1 \leq a \leq b \leq 2 \quad \text{or} \quad 3 \leq a \leq b \leq 4, \\ \xi'_{a,b} - \xi'_{b,a}, & \quad (a, b) = (1, 2) \quad \text{or} \quad (3, 4), \\ \xi_{a,b} - \xi''_{a,b}, & \quad 1 \leq a \leq 2 \quad \text{and} \quad 3 \leq b \leq 4, \\ \xi'_{a,b} + \xi'_{b,a}, & \quad 1 \leq a \leq 2 \quad \text{and} \quad 3 \leq b \leq 4. \end{aligned} \quad (7)$$

Moreover

$$\begin{aligned} d\mu(\xi_{a,b} + \xi''_{a,b}) &= i(z_a \partial_b + z_b \partial_a + \delta_{a,b}), \\ d\mu(\xi'_{a,b} - \xi'_{b,a}) &= z_a \partial_b - z_b \partial_a, \\ d\mu(\xi_{a,b} - \xi''_{a,b}) &= -i(\partial_a \partial_b + z_a z_b), \\ d\mu(\xi'_{a,b} + \xi'_{b,a}) &= \partial_a \partial_b - z_a z_b. \end{aligned} \quad (8)$$

Let $\mathrm{MU}(2, 2)$ denote the inverse image of $\mathrm{U}(2, 2)$ in $\mathrm{Mp}(4; \mathbf{R})$. Then

$$\nu = \det^{1/2} \otimes \mu|_{\mathrm{MU}(2,2)} \quad (9)$$

is a well-defined unitary representation of $\mathrm{U}(2, 2)$. There, it agrees with μ as given in (8), except for the cases $a = b$ in the first line, which become

$$\begin{aligned} d\nu \begin{pmatrix} ix_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & ix_4 \end{pmatrix} &= \frac{1}{2} d\nu \left\{ \sum_1^2 x_a (\xi_{a,a} + \xi''_{a,a}) - \sum_3^4 x_b (\xi_{b,b} + \xi''_{b,b}) \right\} \\ &= i \left\{ \sum_1^2 x_a (z_a \partial_a + 1) - \sum_3^4 x_b (z_b \partial_b) \right\}. \end{aligned} \quad (10)$$

We know [4, Theorem 4.23] that $\mathcal{H}$ is the direct sum of subspaces

$$\mathcal{H}_d = \text{closed linear span of } \{\varphi_n: n_1 + n_2 - n_3 - n_4 = d\} \quad (11)$$

and that

$$\nu = \bigoplus_{d=-\infty}^{\infty} \nu_d, \quad (12)$$

where ν_d represents $\mathrm{U}(2, 2)$ irreducibly on $\mathcal{H}_d$.

In this note, we are concerned with $\nu_0|_{\mathrm{SU}(2,2)}$.

3. Restriction to $S(U(2) \times U(2))$. $S(U(2) \times U(2))$ is a maximal compact subgroup of $SU(2, 2)$, and consists of matrices $g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $A, B \in U(2)$ and $(\det A)(\det B) = 1$. Its center Z consists of those g with $A = e^{i\theta}I_2$ and $B = e^{-i\theta}I_2$, while its derived group consists of those g with $A, B \in SU(2)$. The kernel of $\alpha: SU(2, 2) \rightarrow SO(2, 4)$ is just $\{\pm I_4\}$. Moreover, α maps $S(U(2) \times U(2))$ onto $SO(2) \times SO(4)$, Z onto $SO(2)$, and $SU(2) \times SU(2)$ onto $SO(4)$.

Now the natural action of $SO(4)$ on S^3 lifts to $T^+(S^3)$ and so to X . By [8] and [1], the action of $g \in SO(4)$ on $\mathcal{H}_G$ sends f to $z \mapsto f(g^{-1} \circ z)$. Hence $\mathcal{H}_G$ is a direct sum of subspaces

$$(\mathcal{H}_G)_k = \text{span of } \{f|_X: f \text{ homogeneous polynomial of degree } k\}, \quad (13)$$

and

$$\begin{aligned} \pi_G|_{SO(4)} \text{ preserves } (\mathcal{H}_G)_k \text{ and acts irreducibly on} \\ \text{it by the } (k+1)^2\text{-dimensional representation } \overset{k}{\circ} \otimes \overset{k}{\circ}. \end{aligned} \quad (14)$$

Moreover, results of [8] and [1] prove that

$$(\mathcal{H}_G)_k \text{ is the } i(k+1)\text{-eigenspace of } d\pi_G \left[\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 0_{4,2} & 0_{4,4} \end{array} \right]. \quad (15)$$

This describes $\pi|_{SO(2) \times SO(4)}$ and hence $\pi|_{S(U(2) \times U(2))}$.

On the other hand, let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{u}(2, 2)$ consisting of the diagonal matrices seen in (10), and let ε_j ($j = 1, 2, 3, 4$) be the linear functional

$$\begin{bmatrix} ix_1 & & & \\ & \ddots & & \\ & & ix_4 & \end{bmatrix} \mapsto ix_j \text{ on } \mathfrak{h}.$$

We use the simple root system $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \overset{\alpha_3}{\circ}$ for $\mathfrak{u}(2, 2)$ and $\overset{\alpha_1}{\circ} \oplus \overset{\alpha_3}{\circ}$ for $\mathfrak{u}(2) \oplus \mathfrak{u}(2)$, where $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$. Then [9, Lemma 5.3] $\mathcal{H}_0$ is a direct sum of subspaces

$$\mathcal{H}_{r,r} = \text{span of } \{\varphi_n: n_1 + n_2 = r = n_3 + n_4\} \quad (16)$$

and

$$\begin{aligned} \nu_0|_{U(2) \times U(2)} \text{ preserves } \mathcal{H}_{r,r} \text{ and acts irreducibly on it} \\ \text{by the representation with highest weight } (\varepsilon_1 + \varepsilon_2) + r(\varepsilon_1 - \varepsilon_4). \end{aligned} \quad (17)$$

Hence

$$\begin{aligned} \nu_0|_{SU(2) \times SU(2)} \text{ acts irreducibly on } \mathcal{H}_{r,r} \text{ by the} \\ (r+1)^2\text{-dimensional representation } \overset{r}{\circ} \otimes \overset{r}{\circ}, \end{aligned} \quad (18)$$

and

$$\mathcal{H}_{r,r} \text{ is the } 2i(r+1)\text{-eigenspace of } dv_0 \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix}. \quad (19)$$

Comparing (14) with (18) and (15) with (19), and remembering that

$$d\alpha \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 2 & & \\ & & & 0_{2,4} \\ \hline -2 & 0 & & \\ & & 0_{4,2} & \\ \hline & & & 0_{4,4} \end{array} \right],$$

we have

LEMMA. $\pi_1|_{\mathfrak{S}(U(2) \times U(2))}$ and $\nu_0|_{\mathfrak{S}(U(2) \times U(2))}$ are unitarily equivalent.

4. Equivalence of π_1 and ν_0 . In the simple root system $\{\alpha_1, \alpha_2, \alpha_3\}$ of §3, π_1 and ν_0 have the same lowest highest weight $\varepsilon_1 + \varepsilon_2$. Thus [9, Theorem 5.8] gives our result:

THEOREM. The representations π_1 and ν_0 of $SU(2, 2)$ are unitarily equivalent.

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