SHORTER NOTES

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NONTRIVIALY PSEUDOCOMPLEMENTED LATTICES ARE COMPLEMENTED

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ABSTRACT. Nontrivially pseudocomplemented lattices are complemented.

A lattice L with 0 and 1 is called *pseudocomplemented* if for each $a \in L$ there is a largest a^* with $a \wedge a^* = 0$. Such an a^* is called the pseudocomplement of a . A pseudocomplemented lattice is called *nontrivially pseudocomplemented* if each nonunit element has a nonzero pseudocomplement.

THEOREM. *Nontrivially pseudocomplemented lattices are complemented.*

PROOF. Let L be a nontrivially pseudocomplemented lattice. First observe that the intersection of ultrafilters (maximal dual ideals of some authors) of L is equal to $\{1\}$. Indeed, for any nonunit element a of L , a Zorn's Lemma argument shows the existence of a filter (dual ideal) maximal subject to the conditions (i) containing the (proper) principal ideal $\{x \in L: x \geq a^*\}$ and (ii) not containing a . It is easy to see that this filter is an ultrafilter of L since any larger filter either coincides with it or is equal to L according to whether a belongs to or does not belong to the larger filter. Clearly this ultrafilter does not contain a .

Now let $b \in L$ be arbitrary. We shall show that $b \vee b^* = 1$ or, equivalently, $b \vee b^*$ belongs to all ultrafilters of L and thus b^* is a complement of b . Let U be an arbitrary ultrafilter of L . If $b \in U$ then so is $b \vee b^*$, so we may assume that $b \notin U$. By the maximality of U , the filter generated by b and U contains 0, so there is an x in U with $b \wedge x = 0$. But b^* is the largest element whose meet with b equals zero, so b^* dominates the element x of (the filter) U , showing that it is itself in U . Clearly again $b \vee b^*$ is in U and the rest follows.

Remarks. (1) The nontriviality assumption is essential. For example chains are (trivially) pseudocomplemented but not necessarily complemented.

(2) A complete lattice is called $\wedge$ -distributive if, for any $b \in L$ and any subset A of L , $b \wedge \bigvee A = \bigvee \{b \wedge a: a \in A\}$. A lattice is called weakly

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disjunctive if for each nonunit element a of L there is a nonzero element b with $a \wedge b = 0$. Clearly complete, $\wedge$ -distributive weakly disjunctive lattices are nontrivially pseudocomplemented but the converse is false as seen by considering the five element nonmodular pentagon. In [2] MacNab shows that complete, $\wedge$ -distributive weakly disjunctive lattices are Boolean algebras. The proof in [2] uses distributivity to show that the lattice is complemented. By the above theorem this follows easily without any use of distributivity and gives MacNab's result as a corollary.

(3) In [1] we strengthen a result of Tarski by characterizing complete atomic Boolean algebras as semisimple complete and completely distributive lattices. As pointed out by MacNab in [2], this follows from the theorem in [2] (and so from the theorem here) using Tarski's result. Notice however that the proof in [1] strengthens Tarski's result without using it.

REFERENCES

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