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SELF-INJECTIVE RINGS[1]

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ABSTRACT. In 1958 Matlis proved that the study of Noetherian complete local rings could be subsumed under the study of injective modules E over a commutative ring A such that $B = \mathrm{End}_A E$ is commutative. In this case $B = \mathrm{End}_B E$, and E is said to be *strongly balanced* over B. The main theorem of this paper shows that the study of strongly balanced injectives over any ring, and hence the study of Morita self-dualities, is contained in the study of self-injective rings.

Introduction. Let mod-A (A-mod) denote the category of all right (left) A-modules over a ring A. For a noncommutative ring B and a two-sided B-bimodule E, in a natural way the Cartesian product R is a ring, the so-called *split-null* or *trivial extension of E by B*; also called the *semidirect product* (*ring*) of the bimodule E and denoted by $R = (B, E)$.

THEOREM. *$R = (B, E)$ is an injective (injective cogenerator) in* mod-R *iff E is an injective (injective cogenerator) in* mod-B *such that $B =$* End E_B *canonically.*

This theorem shows that any ring B with an injective bimodule E such that $B = \mathrm{End}\, E_B$ is isomorphic to a factor ring $R/(0, E)$ of a self-injective ring, and also leads to new examples of self-injective rings, notably those which are not injective cogenerator (= PF) rings, or not valuation rings.

When R, or E, is a two-sided injective cogenerator, the theorem is a corollary of a theorem of Müller [23].

PROPOSITIONS. We begin with the main lemma used in the proof of Theorem 2.

1. LEMMA. *Let R be a ring, let E be an ideal which is its own left annihilator, ${}^{\perp}E = \{a \in R \mid aE = 0\} = E$, let $B = R/E$. Then E is canonically a B-bimodule. If*

(1.1) *E is injective as a (canonical) right B-module, and*

(1.2) *$B \approx$* End E_B *canonically,*

then R is right self-injective (= injective in mod-R).

Conversely, if R is right self-injective, then for any ideal A, the left annihilator

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$^\perp A$ is an injective right R/A-module, and $\mathrm{End}^\perp A_{R/A} \approx R/{}^{\perp\perp\perp}A$ *canonically. Thus, in this case, any ideal E satifying $E = {}^\perp E$ satisfies* (1.1) *and* (1.2).

PROOF. Let F be the injective hull of R in mod-R, and let

$$F_1 = \mathrm{ann}_F E = \{x \in F \mid xa = 0, \forall a \in E\}.$$

Then, F_1 is a right B-module, and $E = {}^\perp E$ is an injective right B-module by (1.1). Since every B-submodule of F_1 is an R-submodule, then F_1 is an essential extension of $F_1 \cap R = E$ as an R-module, hence as a B-module, so injectivity of E in mod-B implies that $F_1 = \mathrm{ann}_F E = E$. Thus, if $y \in F$, then $yE \subseteq \mathrm{ann}_F E = E$, so y induces an endomorphism $b \in B' = \mathrm{End}\, E_R = \mathrm{End}\, E_B$. Now every $r \in R$ induces an endomorphism $r_s \in \mathrm{End}\, E_B$ via left multiplication; hence $B = R/{}^\perp E = R/E$ embeds in B' canonically. Since $B \approx B'$ canonically by the assumption (1.2), there exists $r \in R$ such that

$$yx = b(x) = r_s x = rx, \quad \forall x \in E,$$

so

$$(y - r)x = 0, \quad \forall x \in E;$$

hence

$$y - r = c \in \mathrm{ann}_F E = E \subseteq R.$$

Therefore, $y = r + c \in R$, $\forall y \in F$, proving that $F = R$ is injective. In this case, for *any* ideal A, ${}^\perp A$ is an injective right R/A-module (e.g., [**3b**, p. 66, Proposition 12]) and every $b \in \mathrm{End}\, A_R$ is induced by an element $r \in R$; hence $R/{}^\perp A \approx \mathrm{End}\, A_R$. Also, $R/{}^{\perp\perp}A \approx \mathrm{End}^\perp A_R = \mathrm{End}^\perp A_{R/A}$, canonically. Taking $A = E = {}^\perp E$, we have the stated properties (1) and (2).

2. THEOREM. *Let $R = (B, E)$ be the semidirect product of a bimodule E over a ring B. Thus, $a(xb) = (ax)b$ for all $a, b \in B$ and $x \in E$, and in $R = B \times E$ addition is componentwise, and multiplication is defined by*:

(2.1) $(a, x)(b, y) = (ab, ay + xb)$.

(*The ring R is $\approx$ the ring of all 2×2 matrices $\left(\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right)$, with $a \in B$, $x \in E$, under ordinary matrix operations.*) *Then*:

(2.2) *R is right self-injective iff E is injective in* mod-B, *and* $B = \mathrm{End}\, E_B$ *canonically.*

(2.3) *R is a right injective cogenerator in* mod-R (= *R is right PF*) *iff E is an injective cogenerator of* mod-B *satisfying* $B = \mathrm{End}\, E_B$ *canonically.*

(2.4) *Assuming* (2.3), *then R is left PF iff E is an injective cogenerator of* B-mod, *and* $B = \mathrm{End}_B\, E$ *canonically.*

PROOF. (2.2). Identify E with $E_1 = \{(0, x) \mid x \in E\}$ in R, and B with $B_1 = \{(b, 0) \mid b \in B\}$. Clearly, $B \approx B_1 \approx R/E_1$ (under $b \mapsto (b, 0) \leftrightarrow (b, 0) + E_1$), and ${}^\perp E_1$ in R is E_1 if E is a faithful left B-module. Thus, assuming E_B injective and $B = \mathrm{End}\, E_B$, that is, assuming (1.1) and (1.2), we have R is injective by Lemma 1. The converse also comes from Lemma 1.

(2.3). Assume that R is right PF (= pseudo-Frobenius). By [**3a**, p. 148, 3.31], an injective right R-module E is cogenerating iff every simple right R-module embeds in E. Since R is a right injective cogenerator ring by assumption, every simple right R-module $V \hookrightarrow R$. Now, since $J = \operatorname{rad} R$ contains any square-zero (or nilpotent or nil) ideal, then $J \supseteq E_1$; hence $R/J \approx B/\operatorname{rad} B$, and every simple right R-module $V = R/M$ corresponds to a simple right B-module $V' = B/M'$. Since V embeds in R, then V' embeds in R. If $v \in R$ and $v = (b, x) \neq 0$ generates V, then $b = 0 \Rightarrow V \subseteq E$, and $b \neq 0 \Rightarrow \exists (0, y) \neq 0 \in E$ such that $(b, x)(0, y) = (0, by) \neq 0 \in V \cap E$; hence $V \cap E = V \subseteq E$ in both cases. This proves that every simple B-module V' embeds in E. Since E is injective by (2.2), this proves that E is cogenerating in mod-B. Moreover, $B = \operatorname{End} E_B$ via (2.2).

These remarks also suffice for the converse of (2.3), since E cogenerating means every simple B-module V' embeds in E; hence every simple R-module V embeds in E. Thus, if E is an injective cogenerator in mod-B, and $B = \operatorname{End} E_B$, then R is injective by (2.2), hence cogenerating inasmuch as every simple right R-module V embeds in $E_1 = (0, E) \subseteq R$.

Proof of (2.4). Let R be left PF. Since E is an injective cogenerator of mod-B (by assumption (2.3)), then E is faithful as a right B-module (see, e.g., [**3a**, p. 92, II4(a)]); hence $E_1^{\perp} = E_1$ follows, so E_1 is an injective left B-module, where $B = R/E_1$, and it is easy to see that $E \approx E_1$ is actually an injective cogenerator of B-mod: If V is a simple left B-module, then V is a simple left R-module, so $V \subseteq R$. But $E_1 V = 0$, since V is a B-module, so $V \subseteq E_1^{\perp} = E_1$ making E_1 a cogenerator of B-mod (cf. [**3b**, p. 199, Exercise 1]). Conversely, if E is an injective cogenerator of B-mod, and $B = \operatorname{End}_B E$, then by the right-left symmetry of Lemma 1 R is left self-injective, hence cogenerating inasmuch as every simple left B-module V embeds in $E_1 = (0, E) \subseteq R$.

2A. Corollary. *Let $R = (B, E)$ be the semidirect product of a ring B and B-bimodule E. Then: R is cogenerating (both sides) iff E is a strongly balanced injective cogenerator over B (both sides). In this case R is PF (both sides).*

Proof. A ring R is cogenerating on both sides iff R is PF on both sides (see [**10**]). Therefore, Theorem 2 applies.

Since there exist rings which are right cogenerating but not injective (see e.g. [**17**]), then (2.3) shows that E a strongly balanced cogenerator over mod-B does not imply that $R = (B, E)$ is cogenerating. However, a theorem of Faith and Walker (e.g. [**3b**, p. 206, Proposition 24.9]) implies that any semilocal right cogenerating ring is injective. Moreover, if E is strongly balanced and cogenerating on both sides, then every one-sided ideal of R is an annihilator [**22**]. Note: by starting with, e.g., a self-injective ring $B = E$, one obtains another self-injective ring $R = (B, E)$ having B as a factor ring, etc.

Every known example of a right PF ring is left PF. (See **[4a]**, **[4b]** for the background of this problem.)

2B. COROLLARY. *If every right PF ring is left PF, then a bimodule E over a ring B satisfies* (2.3) *iff it satisfies the left-right symmetry* (2.3)′.

PROOF. This follows from (2.4).

Thus, the question is whether right PF $\Rightarrow$ left PF can be reduced to a module-theoretic question. Conceivably a negative answer could be found for the latter for the case when E is some strongly balanced injective cogenerator in mod-B for an integral domain B. Thus, does (2.3) imply the following three conditions?

$$(2.3)' \equiv \begin{cases} (2.3\text{a})' & E \text{ is injective in } B\text{-mod}, \\ (2.3\text{b})' & E \text{ is a cogenerator in } B\text{-mod}, \\ (2.3\text{c})' & B = \operatorname{End}_B E \text{ canonically}. \end{cases}$$

A theorem of Kato **[10]** implies that a right PF ring is left PF iff it is left self-injective, and therefore it suffices to prove or disprove (2.3a)′ and (2.3c)′. Moreover, a theorem of E. A. Walker and the author (see, e.g., **[3b**, p. 206, Proposition 24.9**]**) implies that any finitely generated projective cogenerator over a semilocal ring is injective. Thus, since a right PF ring is semiperfect hence semilocal, then (2.3b)′ implies (2.3a)′; that is, it also suffices to prove or disprove (2.3b)′ and (2.3c)′.

A mapping $f\colon L \to E$ of a left ideal L of B into a B-module E is a *Baer homomorphism* if there exists $m \in E$ such that $f(x) = mx$, $\forall x \in L$. Then E is (FP)-injective in B-mod if every mapping $f\colon L \to E$ from any (finitely generated) left ideal L is a Baer homomorphism. Any right PF ring is left FP-injective (a result which follows from the theorem of Jain **[25]** to the effect that R is left FP-injective iff every finitely presented right R-module is torsionless). Moreover, $R = (B, E)$ is left FP-injective only if E is FP-injective in B-mod, so we conclude that (2.3) implies the latter. Thus, (2.3) does imply some *form* of injectivity of E in B-mod. Actually, left FP-injectivity of (B, E) also implies: (1) that E is finitely quasi-injective in B-mod in the sense of **[26]**, (2) that the right ideals of B satisfy the double annihilator condition with respect to E, and similarly, (3) that the right B-submodules X of E of the form $X = Y + EK$ for a finitely generated right ideal K of B, and finitely generated B-submodule of E in mod-B, also satisfy the double annihilator condition with respect to B. (It would be of obvious interest to characterize FP-injectivity of (B, E).)

3. COROLLARY. *If E is a B-bimodule satisfying* (2.3), *then B is semiperfect, and E is a finite direct sum of indecomposable injectives. Therefore, there are only finitely many nonisomorphic simple B-modules, and E has finite socle.*

PROOF. Since $R = (B, E)$ is right PF, then R is semiperfect, e.g., by Osofsky's theorem [**17**] (cf. [**3b**, p. 213, Theorem 24.32]), and the rest follows from this.

4. THEOREM. *Let B be a commutative Noetherian ring with a strongly balanced injective module E. Then $B = \coprod_{i=1}^{n} B_i$ is a finite product of complete local rings, and $E = \sum_{i=1}^{n} \oplus E_i$, where E_i is the smallest injective cogenerator of B_i, $i = 1, \ldots, n$. Thus, E is the smallest injective cogenerator of B.*

PROOF. Since B is Noetherian, E is a finite coproduct $E = \coprod_{i=1}^{n} E_i$ of indecomposable injectives. Since each E_i has local endomorphism ring, the finite Krull-Schmidt theorem holds, and so B is a semilocal ring, idempotents lift modulo radical (see [**3b**, p. 45, 18.26]), $B = \coprod_{i=1}^{n} B_i$, where $B_i = e_iBe_i \approx \mathrm{End}_B\, E_i$ is a local ring, and $e_i^2 = e_i \in B$ is the projection idempotent, $i = 1, \ldots, n$. Hence, we may assume E is indecomposable and B local. By Matlis' theorem [**13**], in order that B be complete it is necessary and sufficient to show that E is the injective hull of $V = B/\mathrm{rad}\, B$. By the Matlis-Utumi theorem, $J = \mathrm{rad}\, B$ is the set of all b such that $bx = 0$ for some $x \neq 0$. Since J is f.g., and E is uniform, then $W = \mathrm{ann}_E\, J \neq 0$. Thus, W is an R/J-module, hence is semisimple (= a direct sum of simples), whence simple by uniformity, so $W \approx R/J \hookrightarrow E$. Then, E is the injective hull of $V = R/J$, as required.

4A. COROLLARY. *If $B = \mathrm{End}_B\, E$ is a commutative local ring with f.g. radical J, and E injective, then $E = E(B/J)$ is the injective hull of B/J. So E is a cogenerator in* mod-B.[2]

PROOF. Same.

4B. COROLLARY. *If the semidirect product ring $R = (B, E)$ of a Noetherian commutative ring B and module E is self-injective, then R is an injective cogenerator, and a finite product of local injective cogenerators.*

PROOF. By Theorem 2, $B = \mathrm{End}_B\, E$ canonically, and E is an injective B-module, so Theorem 3B applies, and the rest is easy.

An application of Theorem 2 and Matlis' theorem [**13**] yields:

4C. THEOREM. *If B is a Noetherian local ring, and $E = E(B/\mathrm{rad}\, B)$ the injective hull, then $R = (B, E)$ is injective iff B is complete. (Then R is PF.)*

A ring R is a *right valuation ring* (VR) iff the right ideals of R are linearly ordered by inclusion. (A *chain ring* is a variant term for VR.)

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[2] If S is a commutative ring with duality, then there exists a (self) duality context ${}_SF_S$ where F is the minimal injective cogenerator (Theorem of B. J. Müller [**23**]; see also Vámos [**20**, Corollary 1.7]). When the radical of S is finitely generated, then Corollary 4A shows that there is just one self-duality. The dualities for commutative S are in 1-1 correspondence with ring automorphisms of S of order < 2 (Morita [**15**]; cf. [**3b**, p. 199, 23.35]). For other dualities, consult [**1**], [**3b**], [**7**], [**13**]–[**16**], [**20**], [**21**], [**23**], [**24**].

5A. PROPOSITION. *A semidirect product ring $R = (B, E)$ is a right VR iff B is a right VR, E is uniserial, and $bE = E$, $\forall 0 \neq b \in B$.*

PROOF. If R is a right VR, then $B \approx R/(0, E)$ is a right VR, and $E \approx (0, E)$ is uniserial. If $b \neq 0 \in B$, then $(b, 0)R \not\subset (0, E)$; hence

$$(b, 0)R = (bB, 0) + (0, bE) \supseteq (0, E),$$

so $bE = E$. The converse follows by reading up.

A VD is a domain which is a VR. For simplicity, from here on we shall assume that B whence R is commutative.

5B. COROLLARY. *Let E be a faithful B-module. Then $R = (B, E)$ is a VR iff B is a VD and E is a uniserial divisible B-module.*

PROOF. Immediate.

5C. COROLLARY. *Let E be a torsion free module over a domain B. Then $R = (B, E)$ is a VR iff B is a VD and E is a uniserial injective B-module. In this case R is injective iff E is strongly balanced.*

PROOF. Any torsion free divisible module over a domain is injective, so apply the corollary. (Conversely, any injective module is divisible.) The last sentence follows from Theorem 2.

6A. THEOREM. *Let $R = (B, E)$ be a semidirect product ring. The f. a. e.:*

(1) *R is a PFVR (= a VR which is PF).*

(2) *B is an almost maximal valuation domain (AMVD), $E = E(B/\mathrm{rad}\, B)$ is the injective hull of $B/\mathrm{rad}\, B$, and $B = \mathrm{End}_B\, E$.*

(3) *B is a local domain such that $E = E(B/\mathrm{rad}\, B)$ is uniserial and strongly balanced.*

(4) *B is an MVD and $E = E(B/\mathrm{rad}\, B)$ is strongly balanced.*

PROOF. By Gill's theorem **[5]**, a local ring B is AMVR iff $E(B/J)$ is uniserial, where $J = \mathrm{rad}\, B$. Thus, using Theorems 2 and 5A, (2)$\Leftrightarrow$(3) follows. Moreover, (1)$\Leftrightarrow$(3) by 5C and Corollary 4A, and (2)$\Leftrightarrow$(4) by a theorem of Vámos **[19]**.

6B. COROLLARY. *If B is a Noetherian local domain, and $E = E(B/J)$, then the semidirect product ring $R = (B, E)$ is an injective VR iff B is a complete discrete valuation domain. In this case R is PF.*

PROOF. Follows from 6A and Matlis' theorem **[13]** (since B is a Noetherian VD).

7. EXAMPLE. *A noncongenerating injective local ring.* (Levy **[11]**.) Let F be a field, x an indeterminate, and W the family of all well-ordered sets of nonnegative real numbers. Let A denote the ring of all formal power series $\Sigma_{a \in w} c_a x^a$, where $c_a \in F$ and $w \in W$ with the usual addition and multiplication. The proper ideals of A are: the principal ideals

$$(x^b) = \{x^b u | u \in A\}$$

and the ideals

$$(x^{>b}) = \{x^c u | u \in A^* \cup \{0\}, c > b\}$$

where A^* = units of A. (In particular, rad $A = (x^{>0})$.) Levy [**11**] proved that every proper factor ring is self-injective. Now $R = A/(x)$ does not contain a minimal ideal, hence R is injective but not PF. [This corrects a statement of p. 216 of [**3b**] to the effect that every proper factor ring of A is PF! If every factor ring of a ring R is PF (= R is CPF), then R must be Artinian. (See for example [**3b**, p. 238, Proposition 25.4.6A].) However, no factor ring $R = A/I$, where $I \neq \operatorname{rad} A$, can be Artinian, since $(\operatorname{rad} A)^2 = (\operatorname{rad} A) \Rightarrow (\operatorname{rad} R)^2 = (\operatorname{rad} R)$.]

Any infinite product of self-injective rings is self-injective, but never PF since never semiperfect, yielding additional examples of noncogenerating injective rings.

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ADDED JULY 1978. H. Sekiyama informs me that [**28**] contains the characterization of when $R = (B, E)$ is injective (Corollary 4.36). In [**27**] he characterizes i.a. when R is QF-3.

BIBLIOGRAPHY

1. G. Azumaya, *A duality theory for injective modules*, Amer. J. Math. **81** (1959), 249–278.

2. ______, *Completely faithful modules and self-injective rings*, Nagoya Math. J. **27** (1966), 697–708.

3a. C. Faith, *Algebra: Rings, modules, and categories*. I, Die Grundlehren der math. Wissenschaften, Springer-Verlag, Berlin and New York, 1973.

3b. ______, *Algebra*. II: *Ring theory*, Die Grundlehren der math. Wissenschaften, Springer-Verlag, Berlin and New York, 1976.

4a. ______, *Injective cogenerator rings and a theorem of Tachikawa*, Proc. Amer. Math. Soc. **60** (1976), 25–30.

4b. ______, *Injective cogenerator rings and a theorem of Tachikawa*. II, Proc. Amer. Math. Soc. **62** (1977), 15–18.

5. D. T. Gill, *Almost maximal valuation rings*, J. London Math. Soc. (2) **4** (1971), 140–146.

6. M. Ikeda and T. Nakayama, *Some characteristic properties of quasi-Frobenius and regular rings*, Proc. Amer. Math. Soc. **5** (1954), 15–19.

7. A. Jategoankar, *Certain injectives are artinian*, Lecture Notes in Math., vol. 545, Springer-Verlag, Berlin and New York, 1976, pp. 128–139.

8. I. Kaplansky, *Maximal fields with valuation*, Duke Math. J. **9** (1942), 303–321.

9. ______, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc. **72** (1952), 327–340.

10. T. Kato, *Self-injective rings*, Tôhoku Math. J. **19** (1967), 485–495.

11. L. S. Levy, *Commutative rings all of whose homomorphic images are self-injective*, Pacific J. Math. **18** (1966), 149–153.

12. L. S. Levy and G. B. Klatt, *Pre-self-injective rings*, Trans. Amer. Math. Soc. **122** (1969), 407–419.

13. E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528.

14. ______, *Injective modules over Prufer rings*, Nagoya J. Math. **15** (1959), 57–69.

15. K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142.

16. T. Onodera, *Über Kogeneratoren*, Arch. Math. (Basel) **19** (1968), 402–410.

17. B. L. Osofsky, *A generalization of quasi-Frobenius rings*, J. Algebra **4** (1966), 373–387; MR **34** # 4305; erratum; MR **35** # 6443.

18. B. Stenstrom, *Coherent rings and FP-injective modules*, J. London Math. Soc. **2** (1970), 323–329.

19. P. Vámos, *Classical rings*, J. Algebra **34** (1975), 114–129.

20. ______, *Rings with duality*, Proc. London Math. Soc. (3) **35** (1977), 275–289.

21. ______, *Semilocal Noetherian PI-rings*, Bull. London Math. Soc. **9** (1977), 251–256.

22. C. Faith, *FP-injective rings and modules*, in preparation.

23. B. J. Müller, *On Morita duality*, Canad. J. Math. **21** (1969), 1338–1347.

24. M. Upham, *Localization, completion and duality in HNP rings*, Ph.D. Thesis, McGill University, Montreal, 1977.

25. S. Jain, *Flat and FP-injectivity*, Proc. Amer. Math. Soc. **41** (1973), 437–442.

26. V. S. Ramamurthi and K. M. Rangaswamy, *On finitely injective modules*, J. Austral. Math. Soc. **16** (1973), 239–248.

27. H. Sekiyama, *Trivial extension of a ring with balanced condition*, preprint, Inst. Math., Univ. of Tsukuba, Sakaur-Mura Ibaraki, 300-31, Japan, 1978.

28. R. M. Fossum, P. A. Griffith and I. Reiten, *Trivial extensions of abelian categories*, Lecture Notes in Math., vol. 456, Springer-Verlag, Berlin and New York, 1975.

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