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CONTRACTIFICATION OF A SEMIGROUP OF MAPS

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ABSTRACT. Let (X, τ) be a metrizable topological space, $\mathcal{P}(\tau)$ be the family of all metrics on X whose metric topologies are τ. Assume that the semigroup F of maps from X into itself, with composition as its semigroup operation, is equicontinuous under some $d \in \mathcal{P}(\tau)$; then we have the following results:

I. There exists $d' \in \mathcal{P}(\tau)$ such that f is nonexpansive under d' for each $f \in F$.

II. If F is countable, commutative, and for each $f \in F$, there is $x_f \in X$ such that the sequence $(f^n(x))_{n=1}^{\infty}$ converges to x_f, $\forall x \in X$, then there exists $d'' \in \mathcal{P}(\tau)$ such that f is contractive under d'' for each $f \in F$.

III. If there is $p \in X$ such that (1) $\lim_{n\to\infty} f^n(x) = p$, $\forall x \in X$ and $\forall f \in F$, (2) there is a neighbourhood B of p such that $\lim_{m\to\infty} f_{n_1} f_{n_2} \cdots f_{n_m}(B) = \{p\}$ for any choice of $f_{n_i} \in F$, $i = 1, \ldots, m$, and the limit depends on m only, then for each λ with $0 < \lambda < 1$, there exists $d''' \in \mathcal{P}(\tau)$ such that each f in F is a Banach contraction under d''' with Lipschitz constant λ.

1. Introduction. Let (X, d) be a metric space. A map $f: X \to X$ is nonexpansive if $d(f(x), f(y)) \leqslant d(x, y)$ for all $x, y \in X$. If $x \neq y \Rightarrow d(f(x), f(y)) < d(x, y)$, then we say f is Edelstein contractive or simply contractive. Let F be a family of maps on the metric space (X, d). It is evident that F is equicontinuous on X if each f in F is nonexpansive. However if F is equicontinuous on X under d, one cannot claim that even a single map in F is nonexpansive with respect to this metric d. However in the case when F is a semigroup (where the semigroup operation is understood as the composition of maps), we prove that there is a metric d' equivalent to d such that each map in F is nonexpansive under d'. Therefore the notions of equicontinuity and nonexpansiveness of a semigroup of maps are metrically equivalent in the sense that both notions are compatible under some metric which preserves the original metric topology. Furthermore, if F is countable and commutative such that for each $f \in F$ the iterate sequence $\{f^n(x)\}$ converges to the same point for every $x \in X$, then there is a metric d'' equivalent to d such that each f in F is Edelstein contractive with respect to d''. Finally for a not necessarily count-

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able nor commutative semigroup of maps F, under an additional condition, we prove that there is a metric d''' equivalent to d such that each f in F is a Banach contraction with respect to d''' with the same Lipschitz constant.

2. The contractification of a countable commutative semigroup of maps. For a metrizable topological space (X, τ), let $\mathcal{P}(\tau)$ be the family of all metrics on X whose metric topologies are τ.

THEOREM 1. *Let (X, τ) be a metrizable topological space and let F be a semigroup of maps from X into itself. If for some $d \in \mathcal{P}(\tau)$, F is equicontinuous on X under d, then there exists $d' \in \mathcal{P}(\tau)$ such that f is nonexpansive under d' for each f in F.*

PROOF. Without loss of generality, we may assume that the identity map I is in F. Since F is also equicontinuous under the metric $1 \wedge d$, $((1 \wedge d)(x, y) = \min\{1, d(x, y)\})$ and $1 \wedge d \in \mathcal{P}(\tau)$, we may assume that d is a bounded metric. Define

$$d'(x, y) = \sup\{d(f(x), f(y)): f \in F\} \quad \text{for } x, y \in X.$$

It is easy to see that d' is a metric on X. If $g \in F$, we see that

$$\begin{aligned} d'(g(x), g(y)) &= \sup\{d(f(g(x)), f(g(y))): f \in F\} \\ &\leqslant \sup\{d(h(x), h(y)): h \in F\} = d'(x, y), \end{aligned}$$

so that g is nonexpansive with respect to d' for each $g \in F$. It remains to show that d' is equivalent to d. As $d'(x, y) \geqslant d(x, y)$ for any $x, y \in X$, it suffices to show that for any sequence $(x_n)_{n=1}^{\infty}$ in X and $x \in X$, $d(x_n, x) \to 0$ as $n \to \infty$ implies $d'(x_n, x) \to 0$ as $n \to \infty$. Indeed, let $\varepsilon > 0$ be arbitrarily given. Since F is equicontinuous at x, there exists $\delta > 0$ such that

$$d(y, x) < \delta \Rightarrow d(f(y), f(x)) < \varepsilon/2, \quad \forall f \in F. \qquad (*)$$

Since $d(x_n, x) \to 0$ as $n \to \infty$, there is a positive integer N such that $d(x_n, x) < \delta$, $\forall n \geqslant N$. But then for each $n \geqslant N$, $\exists f_n \in F$ such that

$$d'(x_n, x) \leqslant d(f_n(x_n), f_n(x)) + \varepsilon/2. \qquad (**)$$

It follows from $(*)$ and $(**)$ that $\forall n \geqslant N$,

$$d'(x_n, x) \leqslant d(f_n(x_n), f_n(x)) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $d'(x_n, x) \to 0$ as $n \to \infty$

In particular, if we take F in Theorem 1 to be the iterates $\{f^n: n = 1, 2, \dots\}$ of a single map f, then we have:

COROLLARY 1.1. *Let (X, τ) be a metrizable topological space and let f be a map on X to itself. If for some $d \in \mathcal{P}(\tau)$ the iterate sequence of maps $(f^n)_{n=1}^{\infty}$ is equicontinuous under d, then there exists $d' \in \mathcal{P}(\tau)$ such that f is nonexpansive under d'.*

THEOREM 2. *Let (X, τ) be a metrizable topological space and let F be a semigroup of maps from X into itself which is equicontinuous under some $d \in \mathcal{P}(\tau)$. Let $f \in F$ be such that*

(1) *f commutes with each g in F,*

(2) *for some $a \in X$, the sequence $(f^n(x))_{n=1}^{\infty}$ converges to a for each $x \in X$.*

Then there exists a metric d' in $\mathcal{P}(\tau)$ such that f is contractive and each g in F is nonexpansive under d'.

PROOF. By Theorem 1 there is a metric $d \in \mathcal{P}(\tau)$ such that each g in F is nonexpansive under d. Now define

$$d'(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n} d(f^n(x), f^n(y)) \quad \text{for } x,y \in X,$$

where f^0 is the identity map on X. Clearly d' is a metric on X and d' is equivalent to d because $d \leqslant d' \leqslant 2d$. Also each $g \in F$ is d'-nonexpansive, since

$$\begin{aligned} d'(g(x), g(y)) &= \sum_{n=0}^{\infty} \frac{1}{2^n} d(f^n(g(x)), f^n(g(y))) \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n} d(g(f^n(x)), g(f^n(y))) \\ &\leqslant \sum_{n=0}^{\infty} \frac{1}{2^n} d(f^n(x), f^n(y)) = d'(x,y), \quad \forall x,y \in X. \end{aligned}$$

To prove that f is contractive with respect to d', let $x,y \in X$ with $x \neq y$. Suppose $d'(f(x), f(y)) = d'(x,y)$. Then by the definition of d', we have

$$d(f^{n+1}(x), f^{n+1}(y)) = d(f^n(x), f^n(y)), \quad \forall n = 0, 1, 2, \ldots.$$

This implies $d(f^n(x), f^n(y)) = d(x,y) \neq 0$, $\forall n = 1, 2, \ldots,$ which contradicts the fact that $f^n(x) \to a$ and $f^n(y) \to a$. Hence f is contractive under d'.

The following definition can be found in [3].

DEFINITION. Let F be a family of maps on a topological space X to a topological space Y. F is evenly continuous on X if for each $x \in X$, each $y \in Y$ and each neighbourhood U of y, there is a neighbourhood V of x and a neighbourhood W of y such that $f(V) \subset U$ whenever $f(x) \in W, f \in F$.

It is clear from the definition that if F is equicontinuous under some $d \in \mathcal{P}(\tau)$, then F is evenly continuous.

COROLLARY 2.1. *Let (X, τ) be a metrizable topological space and f: $X \to X$. Assume*

(1) *there exists $a \in X$ such that the sequence $(f^n(x))_{n=1}^{\infty}$ converges to a for each $x \in X$,*

(2) *the family $\{f^n: n = 1, 2, \ldots\}$ is evenly continuous on X.*

Then there exists $d' \in \mathcal{P}(\tau)$ such that f is contractive under d'.

PROOF. It follows from Theorem 2 and the fact that (1) and (2) imply the equicontinuity of $F = \{f^n: n = 1, 2, \dots\}$ under any $d \in \mathcal{P}(\tau)$ (see [3]).

COROLLARY 2.2. *Let (X, τ) be a metrizable topological space and $f: X \to X$. Assume that the sequence $(f^n(x))_{n=1}^{\infty}$ converges for each $x \in X$, and the family $\{f^n: n = 1, 2, \dots\}$ is evenly continuous. Then the following are equivalent:*

(1) *there exists $d' \in \mathcal{P}(\tau)$ such that f is contractive under d';*

(2) *there exists $a \in X$ such that the sequence $(f^n(x))_{n=1}^{\infty}$ converges to a for each $x \in X$;*

(3) *for any nonempty compact f-invariant subset Y of X, $\bigcap_{n=1}^{\infty} f^n(Y)$ is a singleton.*

PROOF. The implication (3) $\Rightarrow$ (2) is trivial under the assumption that $(f^n(x))_{n=1}^{\infty}$ converges for each $x \in X$ and that f is continuous. Corollary 2.1 shows that (2) $\Rightarrow$ (1). To prove (1) $\Rightarrow$ (3), let Y be a nonempty compact f-invariant subset of X. Set $A = \bigcap_{n=1}^{\infty} f^n(Y)$. Then f maps A onto A. As $f(A)$ is compact, there exists $x, y \in A$ such that $d'(f(x), f(y))$ is the diameter $\delta(f(A))$ of $f(A)$ under d'. Suppose $x \neq y$, then

$$\delta(f(A)) = d'(f(x), f(y)) < d'(x, y) \leqslant \delta(A).$$

This contradicts the fact that $f(A) = A$. Hence $\delta(f(A)) = 0$, i.e., A is a singleton.

REMARKS. (i) Note that the assumptions in Corollary 2.2 are equivalent to that $\{f^n: n = 1, 2, \dots\}$ is equicontinuous under any $d \in \mathcal{P}(\tau)$ and $(f^n(x))_{n=1}^{\infty}$ converges for each $x \in X$.

(ii) The assumption that $\{f^n: n = 1, 2, \dots\}$ is evenly continuous in Corollary 2.2 cannot be weakened to the condition that f is continuous without affecting the equivalences of (1), (2) and (3). In fact, the counterexample in [2] shows that (1) and (3) are not equivalent if $\{f^n: n = 1, 2, \dots\}$ is not equicontinuous. This means that the main result in [1] is false. The following example shows that (2) and (3) are not equivalent if $\{f^n: n = 1, 2, \dots\}$ is not equicontinuous.

EXAMPLE. Let X be the set of all integers equipped with discrete topology, and let $X^* = X \cup \{\infty\}$ be the one point compactification of X. Define $f: X^* \to X^*$ as follows: $f(n) = n + 1$, $\forall n \in X$, and $f(\infty) = \infty$. Then f is continuous on X^* such that $f^n(x) \to \infty$ as $n \to \infty$, $\forall x \in X^*$. Note that X^* is metrizable since it is a Hausdorff space with a countable base. It is evident that $\bigcap_{n=1}^{\infty} f^n(X^*) = X^*$. So now we have the continuous map f on the compact metrizable topological space X^* such that (2) of Corollary 2.2 is satisfied but not (3). The crucial point is that the family $\{f^n: n = 1, 2, \dots\}$ is not equicontinuous at the point ∞, although it is equicontinuous at any other point of X^*.

Now we come to our main object of this section.

THEOREM 3. *Let* (X, τ) *be a metrizable topological space, and let* F *be a countable commutative semigroup of maps from* X *into itself. If* F *satisfies the conditions*

(1) *for each* $f \in F$, *there exists* $x_f \in X$ *such that the sequence* $(f^n(x))_{n=1}^{\infty}$ *converges to* x_f, *for all* $x \in X$,

(2) F *is equicontinuous under some* $d \in \mathcal{P}(\tau)$,

then there exists $d' \in \mathcal{P}(\tau)$ *such that* f *is contractive under* d' *for each* $f \in F$.

PROOF. We may write $F = \{f_1, f_2, f_3, \ldots\}$ for it is countable. Apply Theorem 2 to each map f_n; we get a metric $d_n \in \mathcal{P}(\tau)$ such that f_n is contractive under d_n and f_k is nonexpansive under d_n for each $k \geq 1$. Now we have a sequence of metrics d_n in $\mathcal{P}(\tau)$, from which we define

$$d'(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}, \quad \forall x,y \in X.$$

It is clear that d' is a metric on X. By the special property of the real valued function $g(r) = r/(1 + r)$, $r \geq 0$ ($g(r)$ is strictly increasing and $g(r) \to 0 \Leftrightarrow r \to 0$), one can prove that $d' \in \mathcal{P}(\tau)$. Each f_k in F is nonexpansive under d', for

$$\begin{aligned} d'(f_k(x), f_k(y)) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f_k(x), f_k(y))}{1 + d_n(f_k(x), f_k(y))} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)} = d'(x, y). \end{aligned}$$

Next we prove that f_k is contractive under d' for each $k \geq 1$. Suppose on the contrary that some f_k is not contractive under d'. Then there exist x,y in X, $x \neq y$, such that $d'(f_k(x), f_k(y)) = d'(x, y)$. This implies

$$\frac{d_n(f_k(x), f_k(y))}{1 + d_n(f_k(x), f_k(y))} = \frac{d_n(x, y)}{1 + d_n(x, y)}, \qquad \forall n = 1, 2, \ldots,$$

which in turn implies $d_n(f_k(x), f_k(y)) = d_n(x, y)$, $\forall n = 1, 2, \ldots,$ and, in particular, that $d_k(f_k(x), f_k(y)) = d_k(x, y)$. This contradicts the fact that f_k is contractive under d_k. Hence each f_k in F is contractive under d'. This completes the proof.

3. The uniform contractification of a semigroup of maps.

THEOREM 4. *Let* (X, τ) *be a metrizable topological space, let* F *be a semigroup, not necessarily commutative, of maps from* X *into itself, and let* $p \in X$. *Assume that* F *satisfies the following conditions*:

(1) $\lim_{n\to\infty} f^n(x) = p$, $\forall x \in X$ *and* $f \in F$;

(2) *there exists a neighbourhood* B *of* p *such that* $\lim_{m\to\infty} f_{n_1} \cdots f_{n_m}(B) = \{p\}$ *for any choice of maps* $f_{n_i} \in F$, $i = 1, \ldots, m$, *and the limit depends on* m *only*;

(3) *F is equicontinuous under some $d \in \mathcal{P}(\tau)$.*

Then for each λ with $0 < \lambda < 1$, there exists a metric $d' \in \mathcal{P}(\tau)$ such that each $f \in F$ is a Banach contraction with Lipschitz constant λ.

PROOF. Apply Theorem 1 to get a metric $d \in \mathcal{P}(\tau)$ such that each $f \in F$ is nonexpansive under d. By (2) we may assume B to be a d-open ball with centre p and satisfying condition (2). Then B is invariant under each $f \in F$. For each positive integer m, define

(4) $A_0 = \{B\}, A_m = \{f_{n_1} \cdots f_{n_m}(B): f_{n_1}, \ldots, f_{n_m} \in F\},$

$$A_{-m} = \left\{(f_{n_1} \cdots f_{n_m})^{-1}(B): f_{n_1}, \ldots, f_{n_m} \in F\right\}.$$

Let I be the set of all integers. Then we have

(5) A_{m+1} is a refinement of A_m for $m \in I$;

(6) $\bigcup_{m \in I} A_m^0$ is an open covering of X, where $A_m^0 = \{$the interior of A: $A \in A_m\}$;

(7) $\lim_{m\to\infty} d[\bigcup_{A \in A_m} A] = 0$, where $d[\mathcal{C}]$ is the d-diameter of the set $\mathcal{C}$.

It is evident that (5) and (7) follow from (4) and (2), respectively. (6) follows from (1) and the fact that A_m is a family of open sets for $m \leq 0$. For $x,y \in X$, let $\mathcal{S}(x, y)$ be the set of all possible finite sequences $(\langle x_i, n_i\rangle)_{i=0}^m$ in $X \times I$ where $x_0 = x$, $x_m = y$ and for each $i = 1, \ldots, m$, $\{x_{i-1}, x_i\} \subset A$ for some $A \in A_{n_i}$. Let λ, $0 < \lambda < 1$, be given. For $x,y \in X$ we define

$$d'(x, y) = \inf_{\mathcal{S}(x,y)} \sum_{i=1}^{m} \lambda^{n_i} d(x_{i-1}, x_i),$$

where the infimum is taken over all members $(\langle x_i, n_i\rangle)_{i=0}^m$ in the set $\mathcal{S}(x, y)$. As $\mathcal{S}(x, y)$ is nonempty for each $x,y \in X$, d' is well defined on $X \times X$. It is easy to check that d' is a pseudometric on X. We claim that

(8) $d'(f(x), f(y)) \leq \lambda d'(x, y)$, $\forall f \in F$ and $x,y \in X$.

To prove this, let $(\langle x_i, n_i\rangle)_{i=0}^m$ be a member of $\mathcal{S}(x, y)$. Then $(\langle f(x_i), n_i + 1\rangle)_{i=0}^m$ is a member of $\mathcal{S}(f(x), f(y))$. By the definition of d' we have

$$d'(f(x), f(y)) \leq \sum_{i=1}^{m} \lambda^{n_i+1} d(f(x_{i-1}), f(x_i)) \leq \lambda \sum_{i=1}^{m} \lambda^{n_i} d(x_{i-1}, x_i),$$

and hence

$$d'(f(x), f(y)) \leq \lambda \inf_{\mathcal{S}(x,y)} \sum_{i=1}^{m} \lambda^{n_i} d(x_{i-1}, x_i) = \lambda d'(x, y).$$

It remains to prove that d' is a metric and is equivalent to d. To this end we prove

(9) $d' \leq \lambda^n d$ on A_n for all $n \in I$, and

(10) $d \leq \lambda^{-n} d' + d[\bigcup_{A \in A_n} A]$ for each $n \in I$.

One sees that (9) follows from the definition of d' and (10) is equivalent to

(11) $\lambda^n d(x, y) \leq \sum_{i=1}^m \lambda^{n_i} d(x_{i-1}, x_i) + \lambda^n d[\bigcup_{A \in A_n} A]$, $\forall n \in I$, for each member $(\langle x_i, n_i\rangle)_{i=0}^m$ in $\mathcal{S}(x, y)$.

To prove (11), let $(\langle x_i, n_i\rangle)_{i=0}^m$ be a member of $\mathcal{S}(x, y)$ and $n \in I$. We consider the following two cases:

(I) Suppose $x_i \notin A$ for all $i = 0, \ldots, m$ and all $A \in A_n$.

Then from (5) we see that $n_i < n$, $\forall i = 0, \ldots, m$, and hence

(12) $\lambda^n < \lambda^{n_i}$, $\forall i = 0, \ldots, m$. Thus

$$\lambda^n d(x, y) \leq \sum_{i=1}^{m} \lambda^n d(x_{i-1}, x_i) < \sum_{i=1}^{m} \lambda^{n_i} d(x_{i-1}, x_i).$$

Hence (11) holds in this case.

(II) Suppose $x_i \in A$ for some i and some $A \in A_n$.

Let j be the smallest integer and k be the largest integer such that x_j, x_k satisfy statement (II), and let $L = \{1, 2, \ldots, j, k+1, \ldots, m\}$. Then

$$\lambda^n d(x, y) \leq \lambda^n \sum_{i \in L} d(x_{i-1}, x_i) + \lambda^n d(x_j, x_k)$$

$$\leq \sum_{i \in L} \lambda^{n_i} d(x_{i-1}, x_i) + \lambda^n d\Big[\bigcup_{A \in A_n} A\Big] \quad \text{(because (12) holds for } i \in L)$$

$$\leq \sum_{i=1}^{m} \lambda^{n_i} d(x_{i-1}, x_i) + \lambda^n d\Big[\bigcup_{A \in A_n} A\Big].$$

Hence (11) also holds in this case. Therefore (10) is true for all $n \in I$. From (10) we see that if $d'(x, y) = 0$, then $d(x, y) \leq d[\bigcup_{A \in A_n} A]$ for all $n \in I$. Letting $n \to \infty$, we have $d(x, y) = 0$ and hence $x = y$. This shows that d' is a metric on X. To show that d' is equivalent to d, let $(x_k)_{k=1}^{\infty}$ be a sequence in X and $x_0 \in X$. First assume $d(x_k, x_0) \to 0$ as $k \to \infty$. From (6) $\exists n \in I$ such that $x_0 \in$ the interior of A for some $A \in A_n$. As $x_k \to x_0$, $\exists N \in I$ such that $k \geq N \Rightarrow x_k \in A \in A_n$. From (9) we have $k \geq N \Rightarrow d'(x_k, x_0) \leq \lambda^n d(x_k, x_0)$. Hence $d'(x_k, x_0) \to 0$ as $k \to \infty$. Conversely, let $d'(x_k, x_0) \to 0$, as $k \to \infty$. Then from (10) we have

(13) $d(x_k, x_0) \leq \lambda^{-n} d'(x_k, x_0) + d[\bigcup_{A \in A_n} A]$, $\forall n \in I$ and any positive integer k. Given $\varepsilon > 0$, from (7) and $d'(x_k, x_0) \to 0$ as $k \to \infty$, $\exists$ positive integers N, M such that $d[\bigcup_{A \in A_N} A] < \varepsilon/2$ and $d'(x_k, x_0) < (\varepsilon/2)\lambda^N$, $\forall k \geq M$. Then from (13) we have

$$d(x_k, x_0) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall k \geq M.$$

This shows $d(x_k, x_0) \to 0$ as $k \to \infty$. Therefore $d' \in \mathcal{P}(\tau)$ and (8) holds. This completes the proof.

COROLLARY 3.1. *Let* (X, τ) *be a metrizable topological space, F be a semigroup of maps from X into itself, and $p \in X$. Let F satisfy conditions* (1) *and* (2) *of Theorem* 4 *and*

(14) *F is evenly continuous on X, and*

(15) *the closure of the set $F[x] = \{f(x): f \in F\}$ is compact for each $x \in X$.*

Then for each λ with $0 < \lambda < 1$, there exists $d' \in \mathcal{P}(\tau)$ such that each f in F is a Banach contraction under d' with Lipschitz constant λ.

COROLLARY 3.2 ([4, THEOREM (i)]). *Let* (X, τ) *be a metrizable topological space, and let* f *be a continuous map from* X *into itself. Let* $p \in X$ *be fixed. Then for any* $0 < \lambda < 1$, *there exists* $d' \in \mathcal{P}(\tau)$ *such that* f *is a Banach contraction under* d' *with Lipschitz constant* λ *and* $f(p) = p$ *if and only if* $f^n(x) \to p$ *for each* $x \in X$ *and*

(16) $\lim_{n\to\infty} f^n(B) = \{p\}$ *for some neighbourhood* B *of* p.

PROOF. The necessity is obvious. The sufficiency follows from Theorem 4 by observing that if $F = \{f, f^2, \ldots\}$, then (16) is equivalent to (2) in Theorem 4, and furthermore, for a continuous map f with $f^n(x) \to p$, $\forall x \in X$, (16) implies that F is equicontinuous under some $d \in \mathcal{P}(\tau)$ (whose proof can be found in the proof of Theorem (i) in [4]).

REMARK. The main theorem in [5] states:

THEOREM. *Let* (X, τ) *be a metrizable topological space and* f *be a continuous self-map on* X *such that* (i) f *has a fixed point* p *which has an open neighbourhood with compact closure,* (ii) *for every* $x \in X$, *the sequence* $(f^n(x))_{n=1}^{\infty}$ *converges to* p. *Then the following statements are equivalent*

(A) *For each* $\lambda \in (0, 1)$, *there exists* $d_\lambda \in \mathcal{P}(\tau)$, *complete if* X *is topologically complete, such that* f *is a Banach contraction under* d_λ *with contraction constant* λ.

(B) *The sequence of iterates of* f *is evenly continuous.*

The implication (A) $\Rightarrow$ (B) is trivial. We shall show that the implication (B) $\Rightarrow$ (A) can be derived from Corollary 3.2. To see this we prove that (16) holds. Let B be a compact neighbourhood of p (by (i)). We claim that $\lim_{n\to\infty} f^n(B) = \{p\}$. Let $d \in \mathcal{P}(\tau)$. Then from (ii) and (B), we see that $\{f, f^2, \ldots\}$ is equicontinuous under d. For any $\varepsilon > 0$, let $U = B(p, \varepsilon) = \{y \in X: d(y, p) < \varepsilon\}$. For any $x \in B$, $\exists \delta_x > 0$ such that

(17) $d(y, x) < \delta_x \Rightarrow d(f^n(y), f^n(x)) < \varepsilon/2$, $\forall n$.

From (ii), $\exists$ positive integer N_x such that

(18) $n \geqslant N_x \Rightarrow d(f^n(x), p) < \varepsilon/2$.

Now $\{B(x, \delta_x), x \in B\}$ is an open cover of B which is compact, there exist $x_i \in B$, $i = 1, \ldots, n$, such that $B \subset \bigcup_{i=1}^n B(x_i, \delta_{x_i})$. Let $N = \max\{N_{x_i}: i = 1, \ldots, n\}$. We claim that $f^n(B) \subset U$, $\forall n \geqslant N$. Indeed,

$$\begin{aligned} y \in B &\Rightarrow y \in B(x_i, \delta_{x_i}), \quad \text{for some } 1 \leqslant i \leqslant n, \\ &\Rightarrow d(f^n(y), f^n(x_i)) < \frac{\varepsilon}{2}, \quad \forall n, \quad \text{by (17)}, \\ &\Rightarrow d(f^n(y), p) \leqslant d(f^n(y), f^n(x_i)) + d(f^n(x_i), p) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \geqslant N, \quad \text{by (18)}, \\ &\Rightarrow f^n(y) \in U, \quad \forall n \geqslant N. \end{aligned}$$

Hence (16) holds.

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