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ON TWO RESULTS OF J. DUGUNDJI ABOUT EXTENSIONS OF MAPS AND RETRACTIONS

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ABSTRACT. We give a short proof of Dugundji's result that spheres in infinite-dimensional normed spaces are absolute retracts.

In [1], Dugundji proved a very important extension theorem and used it to show the following fact:

THEOREM 1. *Let L be a normed linear space, $B := \{x \in L \mid \|x\| \leqslant 1\}$ and $C := \{x \in L \mid \|x\| = 1\}$. If C is not compact, then C is a retract of B (and hence an absolute retract).*

An important consequence of Theorem 1 for the applications is that C is contractible and an absolute extensor for metrizable spaces (cf. [3]). Another well-known consequence is the fact that closed balls in infinite-dimensional normed spaces do not have the fixed point property for continuous maps.

The aim of this paper is to give a slightly sharper version of the Dugundji extension theorem, which allows a very short and intuitive proof of Theorem 1.

THEOREM 2. *Let X be a metric space, $A \subset X$ a closed subset, D a dense subset of A, and L a normed space (or more generally a locally convex space or an affine space of type m,* cf. [2]). *Then each continuous $f: A \to L$ has a continuous extension $F: X \to L$ with $F(X) \subset f(A) \cup$ [convex hull of $f(D)$].*

The proof of Theorem 2 is the same as the one of the original theorem, if one chooses the points $a_U \in D$ (cf. [2, p. 188]).

PROOF OF THEOREM 1. Since C is not compact, L is infinite-dimensional. Hence L has a proper dense linear subspace L'.[1] By Theorem 2, applied to $X := B$, $A := C$, $D := C \cap L'$ and $f := \mathrm{id}_C$, there exists a continuous map $F: B \to L$ with $F|_C = \mathrm{id}_C$ and $F(B) \subset C \cup$ [convex hull of $C \cap L'$] $= C \cup (B \cap L') \subsetneq B$. Choose $x_0 \in \mathring{B} \setminus F(B)$, and let $r: B \setminus \{x_0\} \to C$ be the radial retraction. Then $r \circ F$ is a retraction from B to C.

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[1]One should recall that it is easy to prove the existence of such a space L': Any unbounded real-valued function on a normalized Hamel basis of L uniquely determines an unbounded linear functional f on L. Then $L' :=$ [kernel of f] is not closed and codim $L' = 1$, and hence L' is dense.

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