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ON p-POWER CENTRAL POLYNOMIALS

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ABSTRACT. We show in this note that if $p^2|n$, p is an odd prime and $UD(Q, n)$ is the generic division algebra of degree n over the rational number field, then for $z \in UD(Q, n)$, z^p central implies z is central.

For S any commutative ring, let $D(S, n, s)$ ($s \geq 2$ always) be the ring of generic $n \times n$ matrices in s variables over S (e.g. [**J**, p. 89]). Set R to be the commutative polynomial ring $S[\{x_{ij}^k | 1 \leq i, j \leq n, 1 \leq k \leq s\}]$. $D(S, n, s)$ can be described as the ring generated over S by the "generic matrices" $X_k = (x_{ij}^k)$, $1 \leq i, j \leq n$, considered as elements of $M_n(R)$. When S is a domain, $D(S, n, s)$ is a domain.

When $S = F$ is an infinite field, $D(S, n, s)$ has a central localization $UD(F, n, s)$, a division ring, called the generic division algebra of degree n over F. Let p be a prime. We say $z \in R$ is p-power central in a ring R if z^p is central but z is not. It is well known that $UD(F, p, s)$ is a crossed product (necessarily a cyclic crossed product) if and only if there is a p-power central z in $UD(F, p, s)$.

Furthermore by, e.g., [**J**, p. 93], $UD(F, p, s)$ is a crossed product if and only if all simple F algebras of degree p over their centers (just degree p for short) are crossed products. Finally, it is clear that a p-power central z exists if and only if such a z exists in $D(F, p, s)$.

Let Z be the ordinary integers. Using a result [**AS**] of Amitsur and this author, we will show that $D(Z, n, s)$ has no p-power central elements when $p^2|n$. We state this as our main theorem.

THEOREM 1. *Suppose $p^2|n$, p is an odd prime, and z^p is in the center of $D(Z, n, s)$ for some $z \in D(Z, n, s)$. Then z is in the center of $D(Z, n, s)$.*

We perform the proof of this theorem with the aid of two lemmas. Let $R = D(Z, n, s)$ and $R_p = D(Z/pZ, n, s)$. There is an obvious natural surjection φ: $R \to R_p$. It is easy to see that there are $s' \in R$, $s \in R_p$ such that $\varphi(s') = s$ and s', s are both images of central polynomials (e.g. [**J**, p. 36]). Set $A = R[1/s']$, $A_p = R_p[1/s]$. By the Artin-Processi theorem (e.g. [**R**, p. 418]) A, A_p are Azumaya algebras. Clearly φ extends to a surjection, also called φ, from A to A_p. As A, A_p are Azumaya, φ is also surjective when restricted to the centers of A, A_p.

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LEMMA 2. *The kernel of* φ: $A \to A_p$ *is* pA.

PROOF. We have inclusion $R \subseteq A \subseteq M_n(Z[x_{ij}^k])[1/s'] =$ (by definition) T and $R_p \subseteq A_p \subseteq M_n(Z/pZ[x_{ij}^k])[1/s] = T_p$. φ clearly extends to a surjection $T \to T_p$ with kernel pT. If I is the kernel of φ, $pT \cap A = I$. Now A/pA is Azumaya and thus embeddable in matrices. By [A, p. 134], $pA = pT \cap A = I$. Q.E.D.

In [AS] Amitsur and this author constructed division algebras of characteristic p and degree p^r, $r \geqslant 2$, with no p-power central elements. We generalize this slightly in order to prove the next lemma.

LEMMA 3. A_p *has no p-power central elements, if* $p^2|n$.

PROOF. Using standard arguments it is enough to construct a division algebra of degree n and characteristic p with no p-power central elements. Let m be such that $p^m|n$ but $p^{m+1}\nmid n$ and set $n' = n/p^m$. From [AS] there is a division algebra, E, of degree p^m and characteristic p with no p-power central elements. The techniques of [AS] easily allow one to assume that F, the center of E, contains a primitive n' root of unity and that there is a cyclic Galois extension K/F of degree n'.

Let $F((x))$ be the field of formal Laurent series in x. $E' = E((x)) = E \otimes_F F((x))$ is a division algebra. If $z \in E'$ is p-power central, we may assume $z = z_0 + z_1x + \cdots$. Thus z_0^p is central in E implying z_0 is central. Continuing by induction we have all z_i are central and thus z is central, a contradiction. Now set $D = E' \otimes_{F((x))} (K((x))/F((x)), x)$, where $(K((x))/F((x)), x)$ is the cyclic algebra as defined, for example, in [J, p. 82]. By [S, p. 166], D is a division algebra of degree n, since $E \otimes_F K$ must be a division algebra. An easy argument using splitting fields shows that if D has a p-power central element, then E' does also. The lemma is proved. Q.E.D.

Returning to the main theorem, it is enough to show that A has no p-power central elements. So assume $z \in A$ is such that z^p is central. As $\bigcap_{n\geqslant 1} p^nA \subseteq \bigcap_{n\geqslant 1} p^nT = (0)$, there is a p^m such that $z \in p^mA - p^{m+1}A$. Considering $p^{-m}z$, we may assume $z \in Z - pA$. Now $\varphi(z)$ is central by Lemma 2, so $z = pz_1 + b_1$ where $b_1 \in C =$ the center of A. Modulo p^3A, $z^p \equiv b_1^p + p^2z_1b^{p-1}$ and so $p^2z_1b_1^{p-1}$ maps to the center of A/p^3A. If $[z_1, A]$ is the set of all $[z_1, a] = z_1a - az_1$ for $a \in A$, then $p^2[z_1, A]b_1^{p-1} \subseteq p^3A$. Since A and A/pA are both domains, we have $[z_1, A] \subseteq pA$. In other words, $z_1 = pz_2 + b_2'$ for $b_2' \in C$. We conclude that $[z, A] \subseteq p^2A$.

Taking the pth power of $z = p^2z_2 + b_2$, $b_2 \in C$, we have that, modulo p^4A, $p^3z_2b_2^{p-1}$ is central. In other words, $p^3[z_2, A]b_2^{p-1} \subseteq p^4A$ implying that $[z_2, A] \subseteq pA$ and so that $[z, A] \subseteq p^3A$. Continuing we have that $[z, A] \subseteq p^mA$ for all m so $[z, A] = \{0\}$. Theorem 1 is proved.

Let K be an infinite field and $p^2|n$. We remark that by well-known isomorphisms, Theorem 1 is equivalent to the nonexistence of an integral polynomial $f(x_1, \ldots, x_m)$ in noncommuting variables such that for any

$a_1, \ldots, a_m \in M_n(K)$, $f(a_1, \ldots, a_m)^p$ is central but for some $a_1, \ldots, a_m \in M_n(K)$, $f(a_1, \ldots, a_m)$ is not central.

As the final remark of this note we observe that Theorem 1 implies that if $p \neq 2$, there are no p-power central elements in $UD(Q, n, s)$, Q the rational numbers.

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