THE INACCESSIBLE INVARIANT SUBSPACES OF CERTAIN C_0 OPERATORS

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ABSTRACT. We extend the Douglas-Pearcy characterization of the inaccessible invariant subspaces of an operator on a finite-dimensional Hilbert space to the cases of algebraic operators and certain C_0 operators on any Hilbert space. This characterization shows that the inaccessible invariant subspaces for such an operator form a lattice. In contrast to D. Herrero's recent result on hyperinvariant subspaces, we show that quasisimilar operators in the classes under consideration have isomorphic lattices of inaccessible invariant subspaces.

Let H be a complex Hilbert space. For T in $B(H)$ (the space of bounded linear operators on H) the set of invariant subspaces for T is given the metric $\text{dist}(M, N) = \|P_M - P_N\|$ where P_M (P_N) is the orthogonal projection on M (N) and " $\| \cdot \|$ " denotes the norm in $B(H)$. An invariant subspace M for T is "cyclic" if there exists x in M such that $\{T^n x\}$ spans M .

R. G. Douglas and Carl Pearcy [3] have characterized the isolated invariant subspaces for T in the case of finite-dimensional H (see [9, Chapters 6 and 7], for the linear algebra used in this article): An invariant subspace M for T is isolated if and only if $M \cap M_i = \{0\}$ or M_i for every noncyclic summand M_i in the primary decomposition for T . In [1] we showed how to view this result as a sharpening of the previously known conditions for the isolation of a solution to a quadratic matrix equation.

The purpose of this note is to extend these results to certain operators on infinite-dimensional spaces. Recall that T in $B(H)$ is "algebraic" if there exists a polynomial p such that $p(T) = 0$. T belongs to " C_0 " if T is a completely nonunitary contraction such that there exists a function f in the Hardy space H^∞ with $f(T) = 0$ in the functional calculus of B. Sz.-Nagy and C. Foiaş [10, Chapter III]. (See [2] for a more recent exposition.) Such a T satisfies a minimal inner function m_T . If m_T is a Blaschke product, $m_T = \prod_{i=1}^\infty f_i^{p_i}$ with

$$f_i(z) = \frac{\bar{\alpha}_i}{|\alpha_i|} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z} \quad (\text{where } |\alpha_i| < 1, \sum (1 - |\alpha_i|) < \infty)$$

for i greater than 1, $f_1(z) = z$, and p_i a nonnegative integer, then the generalized eigenvectors for T span H . Thus T has a "primary decomposition" $H = \bigvee M_i$ where M_i is the kernel of $f_i(T)^{p_i}$ and " $\bigvee$ " denotes a closed linear span.

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In the case of an operator on a finite-dimensional space Douglas and Pearcy showed that the isolated invariant subspaces are the same as the "inaccessible" invariant subspaces, those which do not belong to any nonconstant, continuous curve of invariant subspaces. Herrero [7] was the first to exhibit a nonisolated, inaccessible invariant subspace for an operator on Hilbert space. We shall demonstrate that for invariant subspaces of algebraic operators isolation is equivalent to inaccessibility, but that for C_0 operators with Blaschke product minimal functions inaccessibility is the condition for which the natural analogue of the Douglas-Pearcy characterization holds.

Our main result is

THEOREM 1. *Consider T in C_0 with its minimal function a Blaschke product. An invariant subspace $\mathbf{M}$ for T is inaccessible if and only if the intersection of $\mathbf{M}$ with each noncyclic subspace $\mathbf{M}_i$ in the primary decomposition for T is either $\{0\}$ or $\mathbf{M}_i$. If T is algebraic then such a subspace $\mathbf{M}$ is isolated.*

At the end of this paper we will give the referee's example demonstrating that "inaccessible" may not be replaced by "isolated" in the statement of Theorem 1.

To obtain the interpretation of Theorem 1 for quadratic equations, let $\mathbf{H}_1$ and $\mathbf{H}_2$ be Hilbert spaces with B in $\mathbf{B}(\mathbf{H}_2, \mathbf{H}_1)$, A in $\mathbf{B}(\mathbf{H}_1)$, D in $\mathbf{B}(\mathbf{H}_2)$, and C in $\mathbf{B}(\mathbf{H}_1, \mathbf{H}_2)$. Consider

$$XBX + XA - DX - C = 0$$

for unknown X in $\mathbf{B}(\mathbf{H}_1, \mathbf{H}_2)$. X is called an "inaccessible" solution if it does not belong to any nonconstant, continuous curve of solutions. By an argument almost exactly like that of [1] (a use of characteristic polynomials is easily sidestepped), we deduce:

THEOREM 2. *If K is a solution to (*) and*

$$T = \begin{pmatrix} A + BK & B \\ 0 & D - KB \end{pmatrix}$$

(acting on $\mathbf{H}_1 \oplus \mathbf{H}_2$ in the traditional manner) is a C_0 operator with a Blaschke product minimal function, then the following conditions are equivalent:

- (i) *K is an inaccessible solution to (*);*
- (ii) *each common eigenvalue for $A + BK$ and $D - KB$ has geometric multiplicity one for T .*

We begin the proof of Theorem 1 with

LEMMA 1. *Suppose N in $\mathbf{B}(\mathbf{H})$ satisfies $N^k = 0$. If $\mathbf{M} \neq \mathbf{H}$ is invariant for N and the dimension of $\mathbf{M}$ is not less than k , then $\mathbf{M}$ is an accessible invariant subspace for N .*

REMARK. This result may easily be extended to Banach spaces where the topology on the subspaces is that of the "gap between subspaces" [8].

PROOF OF LEMMA 1. There exists x outside of $\mathbf{M}$ such that Nx belongs to $\mathbf{M}$. Let $\mathbf{S} = \{y_i\}$ be a basis for $\mathbf{N}$, the span of $\{N^i x\}_{i=1}^{k-1}$. There exists z in $\mathbf{M}$ with z outside of $\mathbf{N}$ such that Nz belongs to $\mathbf{N}$. Let $\mathbf{B}$ be a basis for $\mathbf{M}$ containing $\mathbf{S} \cup \{z\}$, and form $\mathbf{B}'$ by deleting z from $\mathbf{B}$. Let $\mathbf{B}_t = \mathbf{B}' \cup \{z + tx\}$ for t a real number. Following [3, Lemma 4.2, p. 336], we define $g(t) = (\mathbf{I} - \mathbf{P}_{\mathbf{K}})(z + tx) / \|(\mathbf{I} - \mathbf{P}_{\mathbf{K}})(z + tx)\|$ where $\mathbf{K}$ is the closed linear span of $\mathbf{B}'$. Then $g(t)$ is continuous and $\|\mathbf{P}_t u - \mathbf{P}_s u\| \leq 2\|u\| \cdot \|g(t) - g(s)\|$, where $\mathbf{P}_t$ ($\mathbf{P}_s$) is the orthogonal projection on the closed linear span of $\mathbf{B}_t$ ($\mathbf{B}_s$). Thus $\mathbf{M}_t = \text{span}(\mathbf{B}_t)$ is a continuous curve of invariant subspaces containing $\mathbf{M}$. $\square$

LEMMA 2. *Let B be a finite Blaschke product and g any inner function which is relatively prime to B . Then there exists u and v in H^∞ such that $Bu + gv = 1$.*

PROOF. Of course Carleson's Corona Theorem [4, p. 202] yields this result, but [10, pp. 134–135] gives an elementary proof of the special case we need in the proof of Theorem 1. $\square$

Our next lemma can be combined with the Corona Theorem in an obvious way to illustrate the extent to which the problem of characterizing the isolated invariant subspaces of an operator in C_0 can be split into the problem solved in this paper and the study of the isolated invariant subspaces of C_0 operators with singular minimal functions. Therefore we refer to this result as a

PROPOSITION. *Suppose T belongs to C_0 and $m_T = f_1 \cdot f_2$ with f_i inner, $i = 1, 2$. Assume the existence of g_i in H^∞ such that $f_1 \cdot g_1 + f_2 \cdot g_2 = 1$. A sequence $\mathbf{N}_n$ of invariant subspaces for T converges to an invariant subspace $\mathbf{M}$ if and only if $\mathbf{N}_n \cap \mathbf{K}_i$ converges to $\mathbf{M} \cap \mathbf{K}_i$ for $\mathbf{K}_i$ equal to the kernel of $f_i(T)$ for $i = 1, 2$.*

REMARK. This proposition is no doubt well known to experts in the theory of C_0 operators because of its simplicity and the existence of the closely related results of P. A. Fuhrmann [5].

To prove the proposition, observe that for all h in $\mathbf{H}$, $h = (f_1 g_1(T) + f_2 g_2(T))h$, so $H = \mathbf{K}_1 + \mathbf{K}_2$. Since f_1 and f_2 are necessarily relatively prime, $\mathbf{K}_1 \cap \mathbf{K}_2 = \{0\}$.

It follows from the open mapping theorem that the natural correspondence C between $\mathbf{H} = \mathbf{K}_1 + \mathbf{K}_2$ and $\mathbf{K} = \mathbf{K}_1 \oplus \mathbf{K}_2$ (external direct sum) implements a similarity between T and an operator S on $\mathbf{K}$. It is easily verified that a similarity between operators induces a homeomorphism between their lattices of invariant subspaces. If $\mathbf{M}$ is an invariant subspace for T , then $m = (f_1 g_1(T) + f_2 g_2(T))m$ for m in $\mathbf{M}$ shows that $\mathbf{M}$ splits: $\mathbf{M} = \mathbf{M} \cap \mathbf{K}_1 + \mathbf{M} \cap \mathbf{K}_2$. Thus if $\mathbf{L}$ and $\mathbf{M}$ are invariant subspaces for T we have

$$\begin{aligned} \text{dist}(C[\mathbf{L}], C[\mathbf{M}]) &= \text{dist}(C[\mathbf{L}] \cap \mathbf{K}_1 \oplus C[\mathbf{L}] \cap \mathbf{K}_2, \\ &\quad C[\mathbf{M}] \cap \mathbf{K}_1 \oplus C[\mathbf{M}] \cap \mathbf{K}_2) \\ &= \max_{i=1,2} \text{dist}(C[\mathbf{L}] \cap \mathbf{K}_i, C[\mathbf{M}] \cap \mathbf{K}_i). \end{aligned}$$

The conclusion of the proposition follows from this equation. $\square$

We are now prepared to complete the proof of Theorem 1.

Suppose $\mathbf{M}$ is an invariant subspace for T satisfying the hypothesis about the primary decomposition of T , $\mathbf{H} = \bigvee_i \mathbf{M}_i$. Recall that a nilpotent operator (or a translation of a nilpotent operator) which has a cyclic vector has a linearly ordered lattice of invariant subspaces, hence each invariant subspace is isolated. It follows that $\mathbf{M} \cap \mathbf{M}_i$ is an isolated invariant subspace for T_i , the restriction of T to $\mathbf{M}_i$, for all i . Suppose $\mathbf{N}_t$ is a continuous curve of invariant subspaces for T with $\mathbf{N}_1 = \mathbf{M}$ ($t \in [0, 1]$). Then for each i , $\mathbf{N}_t \cap \mathbf{M}_i$ converges to $\mathbf{M} \cap \mathbf{M}_i$ as t approaches 1 according to Lemma 2 and the preceding proposition with $B = f_i$ equal to the minimal function for T_i . It also follows that $\mathbf{N}_t \cap \mathbf{M}_i$ is a continuous curve, so $\mathbf{N}_t \cap \mathbf{M}_i = \mathbf{M} \cap \mathbf{M}_i$ for all i . However, $\mathbf{N}_t = \bigvee_i (\mathbf{N}_t \cap \mathbf{M}_i)$ and $\mathbf{M} = \bigvee_i (\mathbf{M} \cap \mathbf{M}_i)$ since the quantities in parentheses are the summands in the primary decompositions for the restrictions of T to $\mathbf{N}_t$ and $\mathbf{M}$ respectively. Thus $\mathbf{N}_t = \mathbf{M}$ for all t . This argument shows that $\mathbf{M}$ is inaccessible. (Because an algebraic operator has only finitely many summands in the primary decomposition, any sequence of invariant subspaces converging to such a subspace $\mathbf{M}$ would necessarily be eventually constant by the preceding argument.)

Now suppose that $\mathbf{M}$ is an inaccessible invariant subspace for T , and $\mathbf{M} \cap \mathbf{M}_i$ equals neither $\{0\}$ nor $\mathbf{M}_i$ for some i . T_i is a translation of a nilpotent operator, so from Lemma 1 we conclude that $\mathbf{M} \cap \mathbf{M}_i$ is finite-dimensional. Let $\mathbf{S}$ be any finite-dimensional invariant subspace of $\mathbf{M}_i$ properly containing $\mathbf{M} \cap \mathbf{M}_i$. $\mathbf{M} \cap \mathbf{M}_i$ is inaccessible in $\mathbf{S}$, so by the Douglas-Pearcy theorem $\mathbf{S}$ must be cyclic for T (because a noncyclic nilpotent operator has no nontrivial inaccessible invariant subspaces). From the fact that every finite-dimensional invariant subspace for T_i is cyclic we deduce that $\mathbf{M}_i$ is a (finite-dimensional) cyclic subspace for T . Thus every inaccessible invariant subspace for T satisfies the hypotheses of Theorem 1. $\square$

It is immediate from Theorem 1 that the inaccessible invariant subspaces form a lattice when m_T is a Blaschke product. D. Herrero [6] has shown that quasisimilarity does not always preserve the lattice of hyperinvariant subspaces for an operator. Specifically, he exhibits two quasisimilar nilpotent operators with different numbers of hyperinvariant subspaces. In contrast, quasisimilar algebraic or C_0 operators with Blaschke product minimal functions have isomorphic lattices of inaccessible invariant subspaces. To verify this statement, suppose that T is such an operator and S in C_0 is quasisimilar to T . Then S has the same minimal function as T [10, p. 125]. All that remains to be checked is that corresponding primary summands for S and T are either both cyclic or both noncyclic. This follows from the fact that a primary summand $\mathbf{M}_i$ for T is cyclic if and only if its dimension is equal to the degree of the minimal function for the restriction of T to $\mathbf{M}_i$.

The referee's example that "inaccessible" may not be replaced by "isolated" in the statement of Theorem 2:

Let $b_1(z) = \prod_{n=1}^{\infty} b(z, a_n)$ be an infinite Blaschke product with distinct zeroes where $b(z, a_n)$ represents the Blaschke factor for the zero a_n . Then for each n , $\sum_{j=0}^{\infty} (\bar{a}_n z)^j$ is a simple eigenvector of the backward shift S^* on H^2 . The corresponding eigenspace H_n forms a positive angle d_n with the space spanned by $\{H_i\}_{i \neq n}$. Choose $\{c_n\}_{n=1}^{\infty}$ so that

(i) the angle between H_n and the subspace spanned by the eigenvector corresponding to c_n is smaller than $2^{-n}d_n$, $n = 1, 2, \dots$;

(ii) $a_n \neq c_n \neq a_m \neq c_m \neq c_n$ for all m and n such that $m \neq n$.

Let $b_0(z) = \prod_{n=1}^{\infty} b(z, c_n)$ and define $T = S^*|(b_0 b_1 H^2)^{\perp}$ where the “ $|$ ” denotes restriction. The minimal function of T is $\overline{b_0(\bar{z})b_1(\bar{z})}$. Now define

$$N_j = (b_1(z)b(z, a_j)^{-1}b(z, c_j)H^2)^{\perp}.$$

$(b_1 H^2)^{\perp}$ is inaccessible in the lattice of invariant subspaces for T by Theorem 1, but $\text{dist}((b_1 H^2)^{\perp}, N_j)$ goes to 0 as j approaches ∞ .

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BIBLIOGRAPHY

1. J. P. Daughtry, *Isolated solutions of quadratic matrix equations*, Linear Algebra and Appl. **21** (1978), 89–94.
2. R. G. Douglas, *Canonical models*, Topics in Operator Theory, Math. Surveys No. 13, Amer. Math. Soc., Providence, R. I., 1974, pp. 161–218.
3. R. G. Douglas and C. Pearcy, *On a topology for invariant subspaces*, J. Functional Analysis **2** (1968), 323–341.
4. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
5. P. A. Fuhrmann, *On sums of Hankel operators*, Proc. Amer. Math. Soc. **46** (1974), 65–68.
6. D. A. Herrero, *Quasimilarity does not preserve the hyperlattice*, Proc. Amer. Math. Soc. **65** (1977), 80–84.
7. ———, *A pathological lattice of invariant subspaces*, J. Functional Analysis **11** (1972), 131–137.
8. D. A. Herrero and N. Salinas, *Analytically invariant and bi-invariant subspaces*, Trans. Amer. Math. Soc. **173** (1972), 117–136.
9. K. Hoffman and R. Kunze, *Linear algebra*, 2nd ed., Prentice-Hall, Englewood Cliffs, N. J., 1971.
10. B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, American Elsevier, New York and Akad. Kiadó, Budapest, 1970.

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